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Computational Complexity in Analysis

SoSe 2015, Exercise Sheet #4

EXERCISE 7:

Recall the definition of (polynomial-time) computable real numbers. Now for $X \subseteq \mathbb{R}$ let

$$\rho^{-1}[X] \quad := \quad \left\{ (a_n)_{n\geq 0} \,:\, a_n \in \mathbb{Z},\ \exists x \in X\ \forall n:\ |x - a_n/2^n| \leq 2^{-n} \right\} \quad \subseteq \quad \mathbb{Z}^{\mathbb{N}}$$

where the Baire space $\mathbb{Z}^{\mathbb{N}}$ is equipped with the metric $d\big((a_n)_n, (b_n)_n\big) = 2^{-\min\{n : a_n \neq b_n\}}$.

a) Prove that the mapping $\rho^{-1}[\mathbb{R}] \ni (a_n)_n \mapsto \lim_n a_n$ is continuous.

b) Prove that $\rho^{-1}[\{x\}]$ is compact for every $x \in \mathbb{R}$; hint: König's Lemma.

c) Prove that $\rho^{-1}[X]$ is compact for every compact $X \subseteq \mathbb{R}$.

EXERCISE 8:

Let (X,d) and (Y,e) be metric spaces. Recall that a modulus of continuity of $f : X \to Y$ is a mapping $\mu : \mathbb{N} \to \mathbb{N}$ such that $d(x,x') \leq 2^{-\mu(n)}$ implies $e\big(f(x), f(x')\big) \leq 2^{-n}$.

a) Construct a polytime bijective $f : [0;1] \to [0;1]$ such that $f^{-1} : [0;1] \to [0;1]$ is *not* polytime.

b) Suppose $f : [0;1] \to [0;1]$ is polytime bijective *and* f^{-1} has a polynomial modulus of continuity. Prove that, then, f^{-1} is again polytime.

A modulus of continuity of f^{-1} is also known as a *modulus of unicity* of f.