Martin Ziegler
Holger Thies

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Computational Complexity in Analysis

SoSe 2015, Exercise Sheet #5

EXERCISE 9:

a) Prove that the function $\varphi : \mathbb{R} \to \mathbb{R}$ with $\varphi(t) := \exp\left(-\frac{t^2}{1-t^2}\right)$ for $|t| < 1$ and $\varphi(t) :\equiv 0$ for $|t| \geq 1$ is continuous and infinitely often differentiable.

b) Prove that $\varphi : \mathbb{R} \to \mathbb{R}$ is polynomial-time computable.

c) Prove that $[0;1] \ni t \mapsto \sum_{N \in \mathbb{N}} \varphi(2tN^2 - 2N)/N^{\ln N}$ is continuous, infinitely often differentiable, and polynomial-time computable.

d) Fix $V \subseteq \mathbb{N}$. Prove:
$f_V : [0;1] \ni t \mapsto \sum_{N \in V} \varphi(2tN^2 - 2N)/N^{\ln N}$ is polynomial-time computable iff $V \in \mathcal{P}$.

EXERCISE 10:

Let $f : [0;1] \to [0;1]$ be polynomial-time computable.

a) Prove that there exists a polynomial-time computable function

$$F :\subseteq \mathbb{Z} \times \mathbb{N} \ni (a, 2^n) \mapsto F(a, 2^n) \in \mathbb{Z} \quad \text{such that}$$
$$\left| f(a/2^n) - F(a, 2^n)/2^n \right| \leq 1/2^n \quad \text{holds whenever} \quad 0 \leq a \leq 2^n \quad . \qquad (1)$$

b) Let F as in Equation (1). Prove that the following language L_F belongs to $\mathcal{NP}$:

$$\left\{ \langle 1^m, \mathrm{bin}(b), \mathrm{bin}(c) \rangle \ : \ \exists a \in \mathbb{Z} : 0 \leq a \leq c : \ F(a, 2^m) \geq b \right\}$$

c) Conclude that $\mathcal{P} = \mathcal{NP}$ implies polynomial-time computability of the function $[0;1] \ni x \mapsto \max\{f(t) : 0 \leq t \leq x\}$. Hint: Exploit that f has a polynomial modulus of continuity.