Martin Ziegler
Holger Thies

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Computational Complexity in Analysis

SoSe 2015, Exercise Sheet #7

The lecture proved the following as equivalent:

i) $\mathsf{FP} = \#\mathsf{P}$

ii) For every polynomial-time computable $h : [0;1] \to \mathbb{R}$, the function $\int h : [0;1] \to \mathbb{R}$ with $x \mapsto \int_0^x h(t)\,dt$ is again polynomial-time computable.

iii) For every smooth (i.e. C^∞) polynomial-time computable $h : [0;1] \to \mathbb{R}$ with support $\operatorname{supp}(f) \subseteq [1/4;3/4]$, $\int h$ is again polynomial-time computable.

EXERCISE 12:
Recall Poisson's partial differential equation

$$\Delta u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = g \tag{1}$$

for some fixed bounded, open, and connected $\Omega \subseteq \mathbb{R}^d$ with boundary $\partial\Omega$ and given $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, where $\Delta u(x_1, \ldots, x_d) = \partial^2_{x_1} u(\vec{x}) + \partial^2_{x_2} u(\vec{x}) + \ldots + \partial^2_{x_d} u(\vec{x})$.

a) In case $d = 1$ with $\Omega = [-1;1]$, verify that the unique solution to Equation (1) is given by

$$u(x) \;=\; \int_{-1}^{x} (x-t) \cdot f(t)\,dt + Cx + C' \;=\; \int_{-1}^{x} \int_{-1}^{t} f(s)\,ds\,dt + Cx + C' \;,$$

where $C := (g_+ - g_- - E)/2$, $C' := (g_+ + g_- - E)/2$, and $E := \int_{-1}^{1} (1-t) \cdot f(t)\,dt = \int_{-1}^{1} \int_{-1}^{t} f(s)\,ds\,dt$.

b) Conclude that, in case $\mathsf{FP} = \#\mathsf{P}$ and whenever f and g_-, g_+ are polynomial-time computable, then so is u.
Conversely if the solution u to Equation (1) is polynomial-time computable for every polynomial-time computable smooth f and g_-, g_+, then $\mathsf{FP} = \#\mathsf{P}$.

c) Employ the following *Maximum Principle* to conclude the uniqueness of a solution to Equation (1):
If $v : \Omega \to \mathbb{R}$ satisfies $\Delta v \equiv 0$, then is attains its maximum on $\partial\Omega$.

d) Let $\Omega_d = \{\vec{x} : \|\vec{x}\|_2 < 1\}$ denote the d-dimensional Euclidean unit ball. Verify that $\int_{\Omega_d} \tilde{G}_d(\|\vec{x} - \vec{y}\|_2)\,d\vec{x}$ exists for every $\vec{y} \in \Omega_d$, where $\tilde{G}_2(r) := -\frac{1}{2\pi} \ln(r)$ and $\tilde{G}_d(r) := \frac{1}{d \cdot (d-2) \lambda(\Omega_d) \cdot r^{d-2}}$ in case $d \geq 3$. Here $\lambda(\Omega_d)$ denotes the volume of Ω_d.