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Computational Complexity in Analysis

SoSe 2015, Exercise Sheet #10

The lecture called a (possibly multivalued and partial) mapping $f :\subseteq \mathbb{R}^\omega \times \mathbb{N} \rightrightarrows \mathbb{R}^\omega \times \mathbb{N}$ *fully polynomial-time computable* iff some Turing machine can, for every $(\bar{x},k) \in \mathrm{dom}(f)$, given any integer sequence (a_m) with $|x_j - a_{\langle j,i\rangle}/2^i| \leq 1/2^i$, produce some $\ell \leq \mathrm{poly}(k)$ and an integer sequence (b_n) within time polynomial in $n+k$ such that $|y_j - b_{\langle j,i\rangle}/2^i| \leq 1/2^i$ holds for some $(\bar{y},\ell) \in f(\bar{x},k)$.

EXERCISE 16:
A discrete *parameterized promise problem* is a pair (A,B) of disjoint subsets of $\{0,1\}^* \times \mathbb{N}$.
(A,B) is *fully polynomial-time* decidable if some Turing machine can decide whether a given $(\vec{x},k) \in A \cup B$ belongs to A or to B within time polynomial in $|\vec{x}|+k$; its behaviour on inputs $(\vec{x},k) \notin (A \cup B)$ is arbitrary.
(A,B) is *fixed-parameter tractable* if the same question can be decided in time bounded by $\psi(k) \cdot n^{\mathcal{O}(1)}$ for some arbitrary $\psi : \mathbb{N} \to \mathbb{N}$.

a) Prove that the following *approximate Knapsack Problem* is fully polynomial-time decidable:

$$\begin{aligned} A &:= \Big\{(w_1,v_1,\dots w_m,v_m,W,V,k) \,\Big|\, \exists J \subseteq \{1,\dots m\} : \textstyle\sum_{j\in J} w_j \leq W \;\wedge\; \sum_{j\in J} v_j \geq V\Big\}, \\ B &:= \Big\{(w_1,v_1,\dots w_m,v_m,W,V,k) \,\Big|\, \forall J \subseteq \{1,\dots m\} : \sum_{j\in J} w_j \leq W \;\Rightarrow\; \sum_{j\in J} v_j < V \cdot (1-\tfrac{1}{k})\Big\} \end{aligned}$$

b) Prove that the following *Vertex Cover Problem* is fixed-parameter tractable:

$$\begin{aligned} A &:= \Big\{(V,E,k) \,\Big|\, (V,E) \text{ undir.graph}, \exists v_1,\dots v_k \in V : \forall e \in E : e \cap \{v_1,\dots v_k\} \neq \emptyset\Big\} \\ B &:= \Big\{(V,E,k) \,\Big|\, (V,E) \text{ undir.graph}, \forall v_1,\dots v_k \in V : \exists e \in E : e \cap \{v_1,\dots v_k\} = \emptyset\Big\} \end{aligned}$$

c) Call $f : \{0,1\}^* \times \mathbb{N} \to \{0,1\}^* \times \mathbb{N}$ computable in *fully polynomial-time* if some Turing machine can, given $(\vec{x},k)$, output $f(\vec{x},k) =: (\vec{y},\ell)$ within a number of steps polynomial in $|\vec{x}|+k$ such that $\ell \leq \mathrm{poly}(|\vec{x}|+k)$.
Suppose (A,B) is a parameterized promise problem and $\big(f[A],f[B]\big)$ is fully polynomial-time decidable. Conclude that (A,B) is fully polynomial-time decidable.

d) Call $f : \{0,1\}^* \times \mathbb{N} \to \{0,1\}^* \times \mathbb{N}$ a *fixed-parameter reduction* if some Turing machine can, given $(\vec{x},k)$, output $f(\vec{x},k) =: (\vec{y},\ell)$ within $\psi(k) \cdot \mathrm{poly}(|\vec{x}|)$ steps such that $\ell \leq \phi(k)$.
Suppose (A,B) is a parameterized promise problem and $\big(f[A],f[B]\big)$ is fixed-parameter tractable. Conclude that (A,B) is fixed-parameter tractable.