Martin Ziegler
Holger Thies

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Computational Complexity in Analysis

SoSe 2015, Exercise Sheet #11

The lecture considered encodings of real numbers, vectors, and sequences by approximations — as well as two different ways of encoding analytic functions $f : [0;1] \to \mathbb{C}$:

i) as complex sequence $\big(f(a/2^m)\big)_{\substack{a=0,\ldots,2^m-1\\ m=1,2,\ldots}}$ together with an integer ℓ such that $\forall z \in \overline{\mathcal{R}}_\ell :$ $|g(z)| \leq 2^\ell$, where g denotes the unique holomorphic extension of f to $\overline{\mathcal{R}}_\ell := \{x + iy : |y| \leq 1/\ell, -1/\ell \leq x \leq 1+1/\ell\}$.

ii) as complex sequence

$$f(0), f(1/k), f(2/k), \ldots, f(1),\ \ f'(0), f'(1/k), \ldots, f'(1),\ \ f''(0), f''(1/k), \ldots f''(1),\ \ldots$$

with an integer k such that the j-th derivative satisfies $\forall x \in [0;1] : |f^{(j)}(x)|/j! \leq 2^{k+j} \cdot k^j$.

EXERCISE 17:

a) Assert that the mapping $[-2^k;+2^k]^\omega \times \mathbb{N} \ni \big((x_j)_j, m\big) \mapsto x_m$ is computable within time polynomial in $n+k+m$.
How long does it take to compute $\big(f(a/2^m)_{a,m}, a, m\big) \mapsto f(a/2^m)$ according to i) ?

b) Describe an algorithm that, given analytic $f : [0;1] \to \mathbb{C}$ in encoding ii) as well as $x \in [0;1]$, produces $f(x)$. Analyze its runtime in terms of the output precision n and the parameter k.

c) Describe an algorithm that, given analytic $f : [0;1] \to \mathbb{C}$ in encoding i) as well as $x \in [0;1]$, produces $f(x)$. Analyze its runtime in terms of the output precision n and the parameter ℓ.
Hint: Employ Cauchy's Differentiation Formula in order to derive a modulus of continuity.