Towards Capturing Order-Independent P

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Descriptive Complexity

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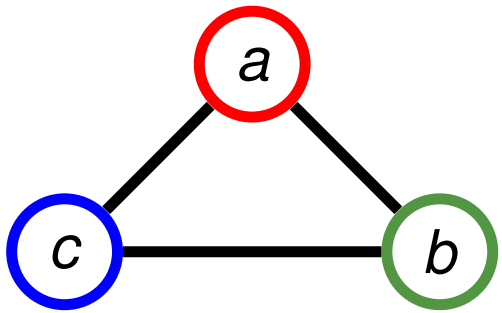
There is a constructive isomorphism between these two approaches.

Think of the Input as a Finite Logical Structure

$$H = (\{a, b, c\}, \leq, E^H, R^H, G^H, B^H)$$

Colored Graph

H



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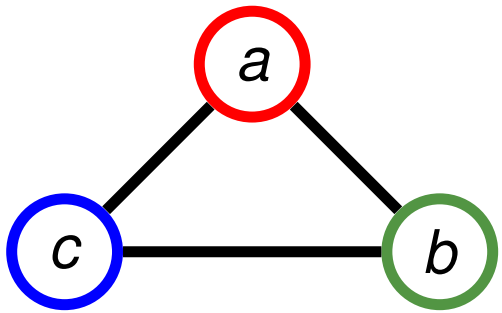

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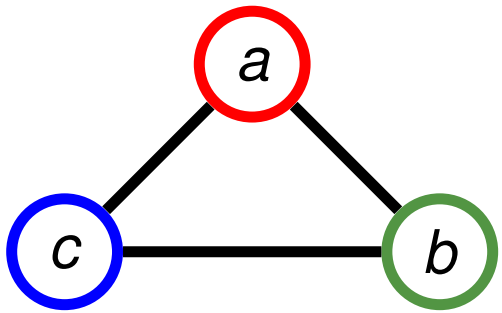

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H		$=$	$(\{a,b,c\}, \leq, E^H, R^H, G^H, B^H)$
	$\leq^H$	$=$	$\{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)\}$
Colored	E^H	$=$	$\{(a,b),(b,a),(b,c),(c,b),(c,a),(a,c)\}$
Graph	R^H	$=$	$\{a\}$
	G^H	$=$	$\{b\}$
	B^H	$=$	$\{c\}$

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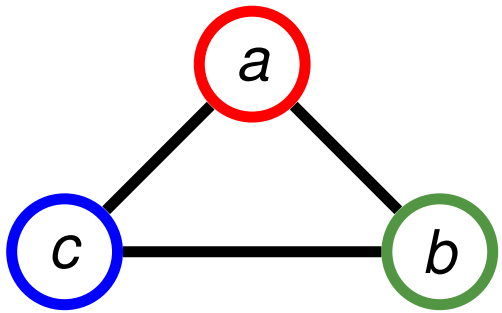

First-Order Logic

input symbols: $E, R, Y, B, \ldots$
variables: $x, y, z, \ldots$
boolean connectives: $\wedge, \vee, \neg$
quantifiers: $\forall, \exists$
numeric symbols: $=, \leq, +, \times, \min, max$

$$\alpha \equiv \forall x \exists y \; E(x, y)$$

$$\beta \equiv \forall xy \; (\neg E(x, x) \; \wedge \; (E(x, y) \rightarrow E(y, x)))$$

$$\gamma \equiv \forall x \; ((\forall y \; x \leq y) \; \rightarrow \; R(x))$$

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In this setting, with the structure of interest being the **finite input**, FO is a **weak** complexity class.

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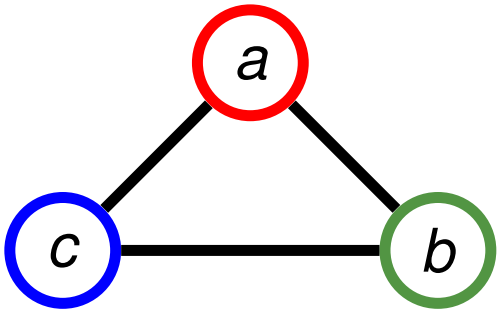
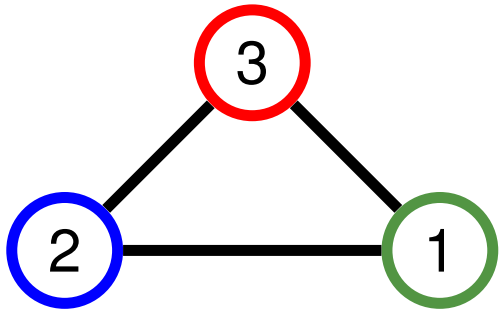
It is **easy** to test if input, H, satisfies α $\quad (H \models \alpha)$.

First-Order Logic

H $\quad a \leq b \leq c$



G $\quad 1 \leq 2 \leq 3$



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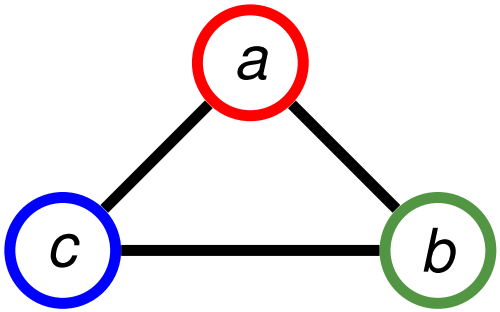

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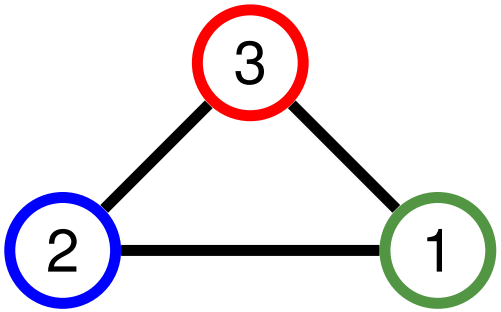

$H \models \alpha \wedge \beta \wedge \gamma$

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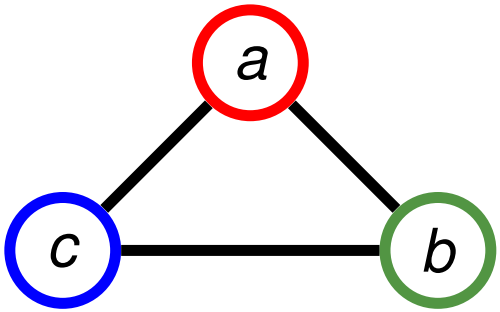

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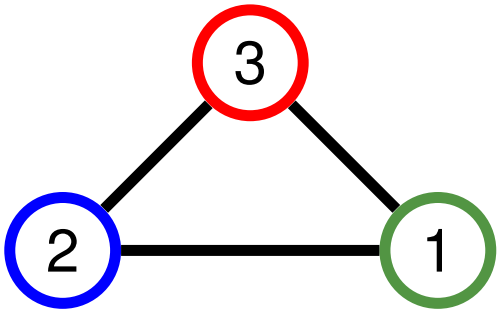

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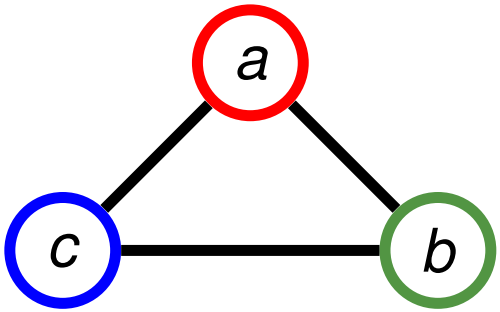

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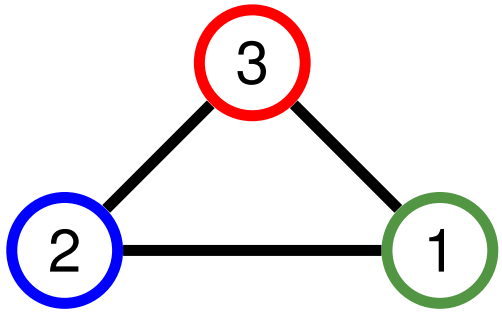

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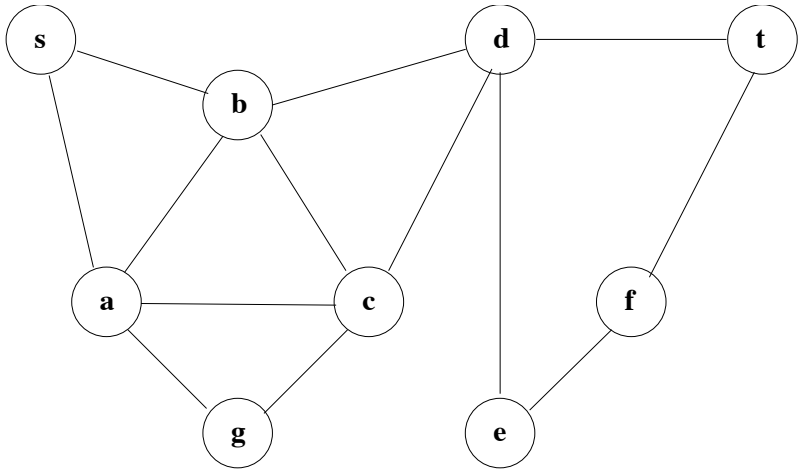
α and β are **order independent**; γ is **order dependent**

Second-Order Logic: FO plus Relation Variables

$$\Phi_{3\text{color}} \equiv \exists R^1\, G^1\, B^1\, \forall x\, y\, ((R(x) \vee G(x) \vee B(x)) \wedge (E(x,y) \rightarrow (\neg(R(x) \wedge R(y)) \wedge \neg(G(x) \wedge G(y)) \wedge \neg(B(x) \wedge B(y)))))$$



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Fagin's Theorem: NP $=$ SO$\exists$

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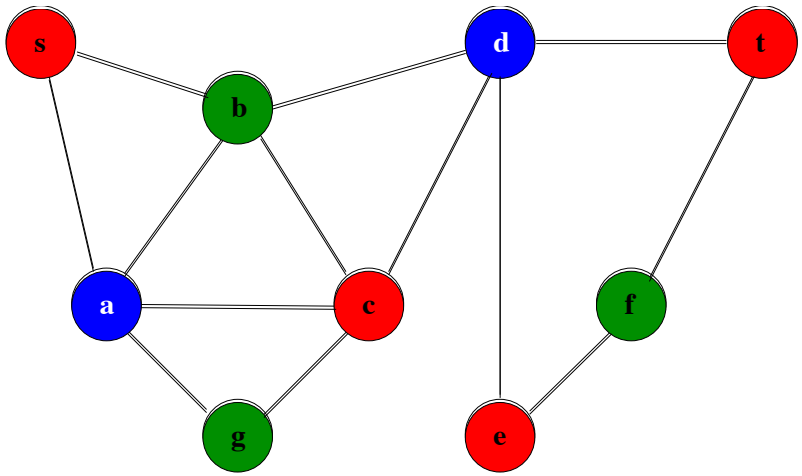

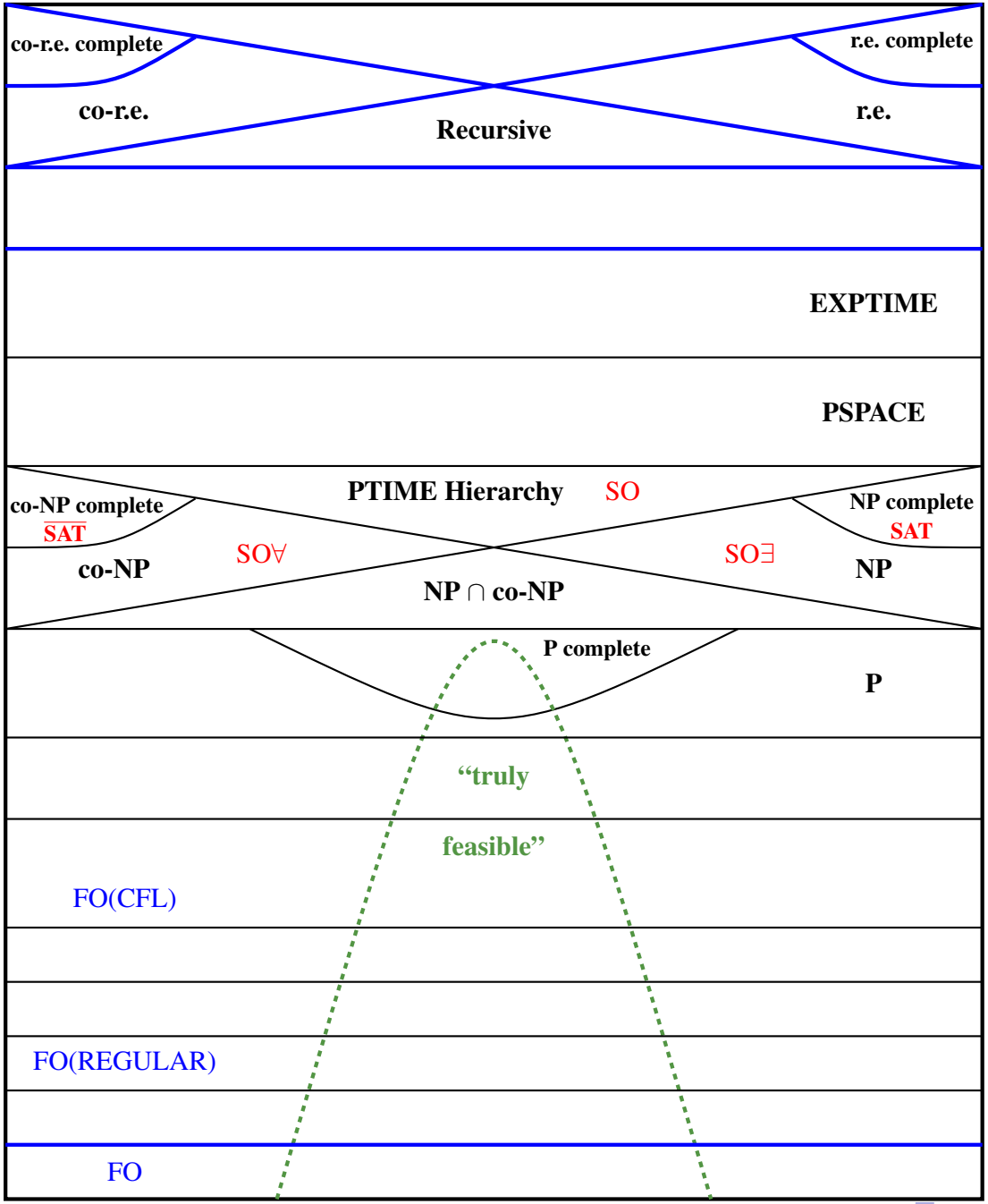
co-r.e. complete
r.e. complete
co-r.e.
Recursive
r.e.
EXPTIME
PSPACE
PTIME Hierarchy
SO
co-NP complete
SAT
NP complete
SAT
SO∀
SO∃
co-NP
NP
NP ∩ co-NP
P complete
P
"truly
feasible"
FO(CFL)
FO(REGULAR)
FO

Inductive Definitions and Least Fixed Point

$$\text{REACH} = \left\{ G, s, t \;\middle|\; s \xrightarrow{\star} t \right\}$$

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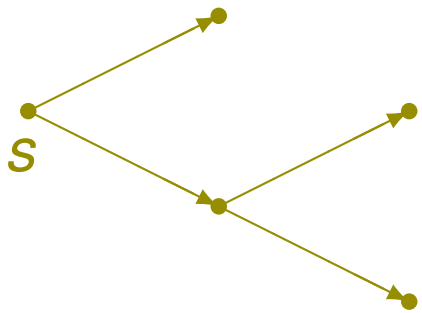

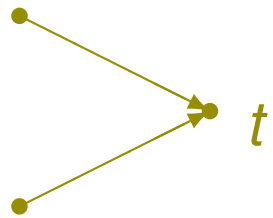

Inductive Definitions and Least Fixed Point

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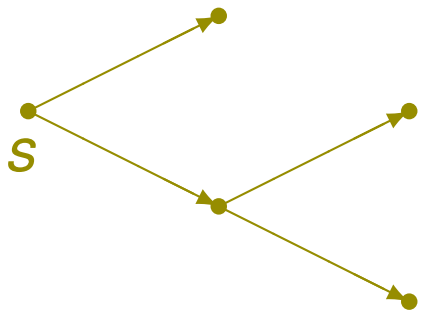

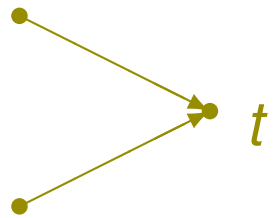

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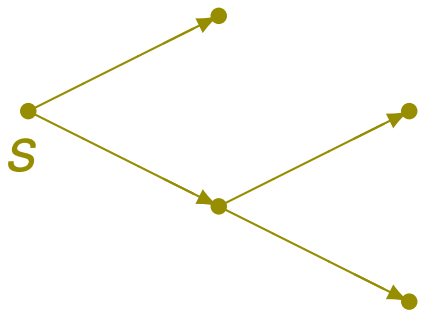

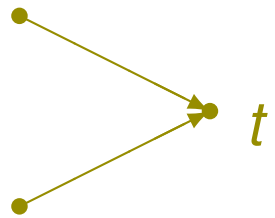

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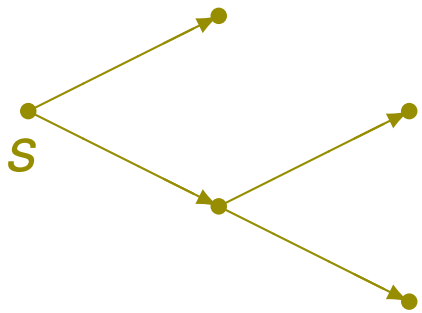

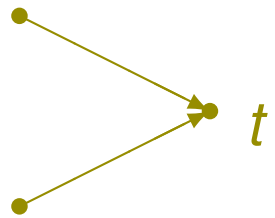

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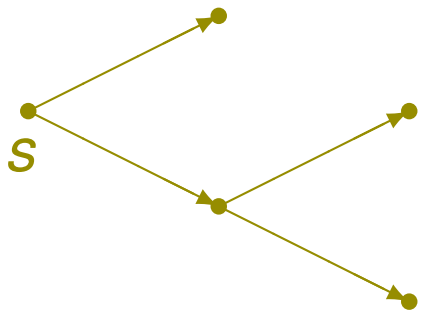

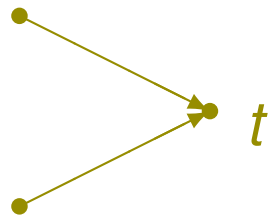

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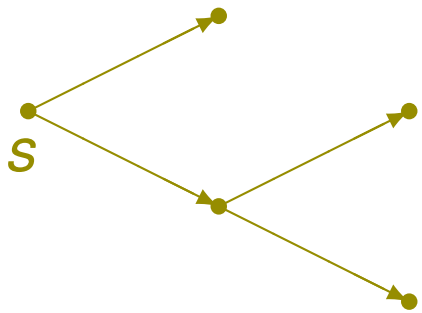

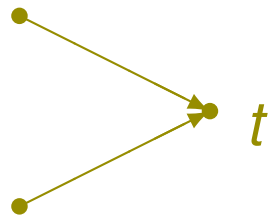

LFP is a Polynomial Iteration Operator

Thm. $\mathrm{P} = \mathrm{FO}(\mathrm{LFP}) = \mathrm{FO}[n^{O(1)}]$

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Wanted: a language capturing Order-Independent P (OIP).

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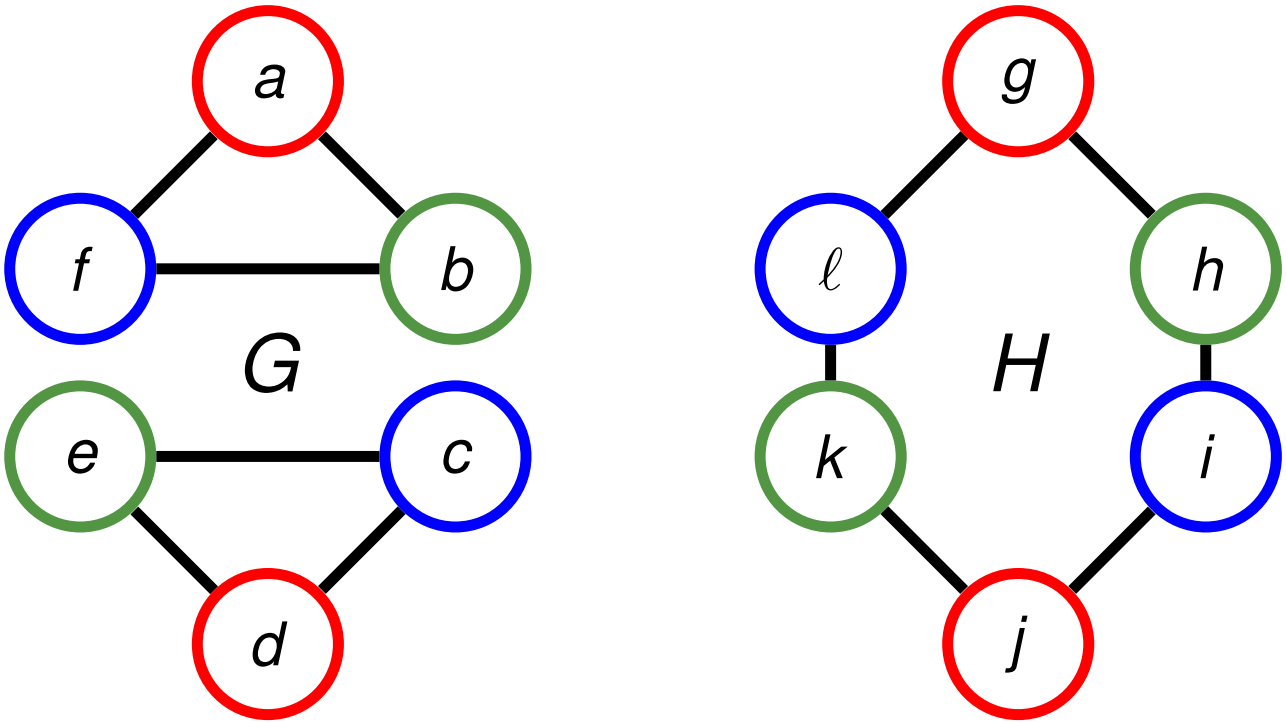
How do we prove $\mathrm{EVEN} \notin \mathrm{FO}(\mathrm{wo}{\leq})(\mathrm{LFP})$?

Ehrenfeucht-Fraïssé Game

$\mathcal{G}_m^k(G, H)$ m moves, k pebbles, 2 players



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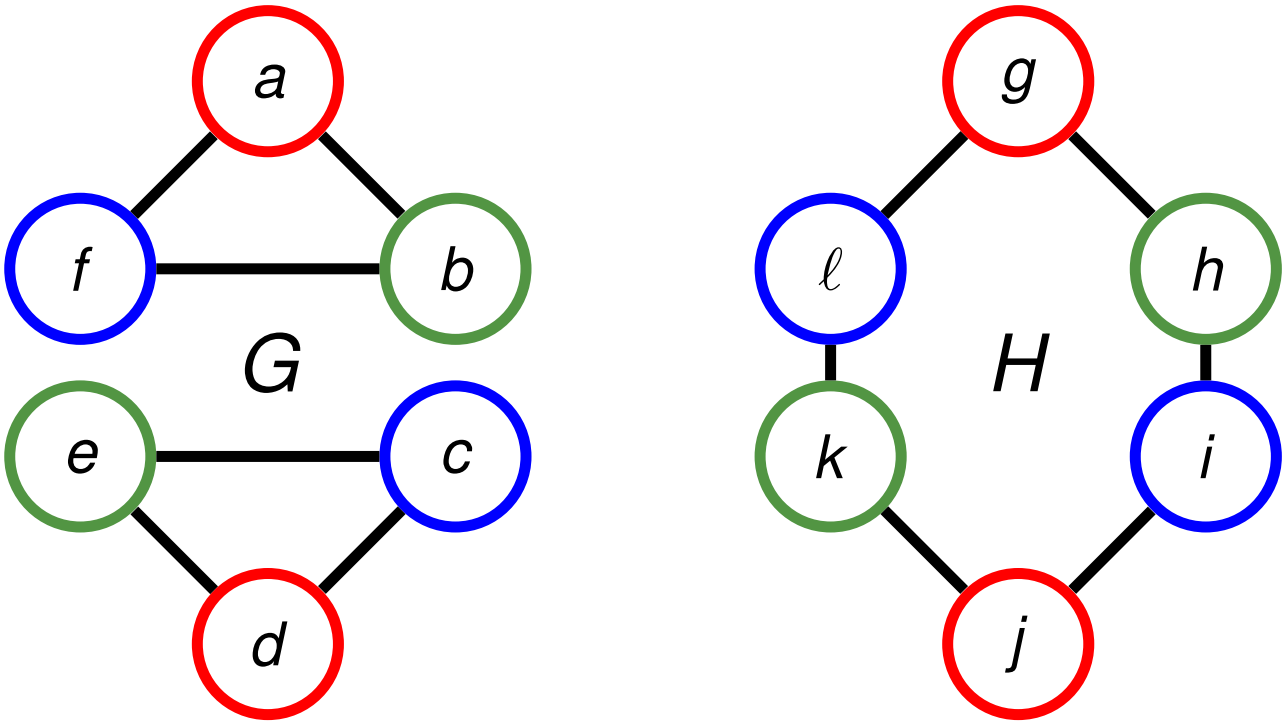

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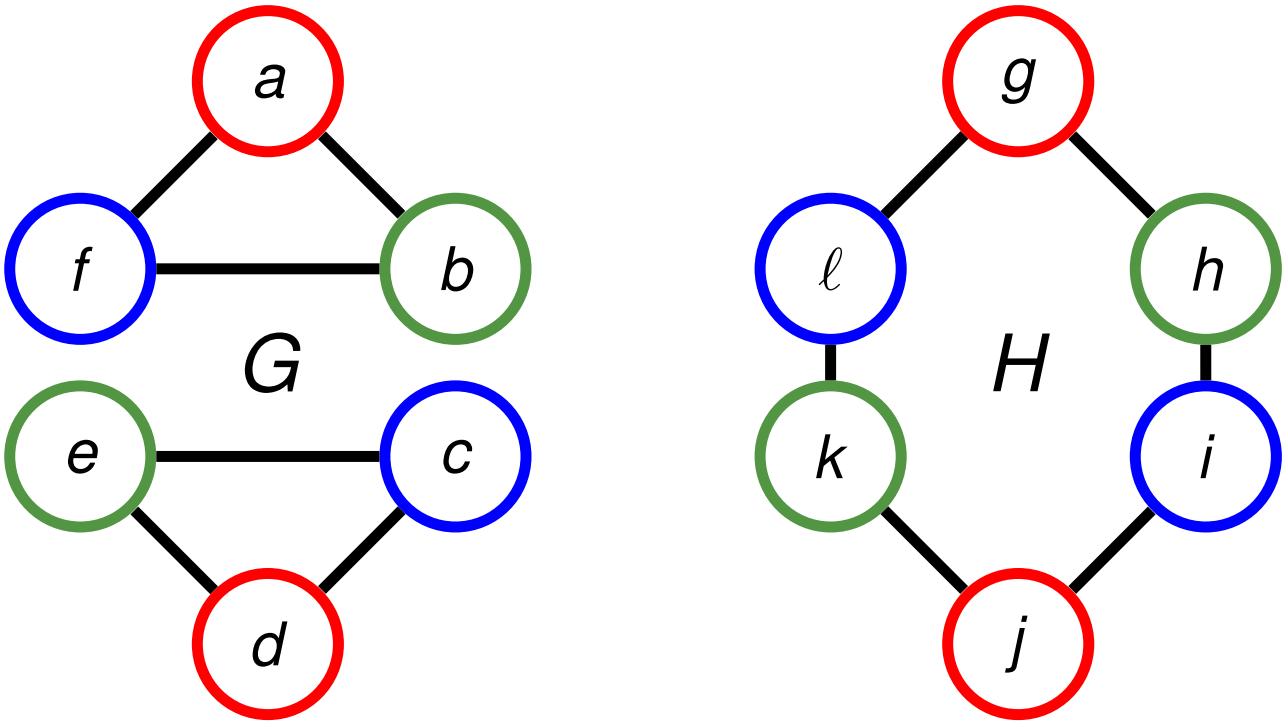

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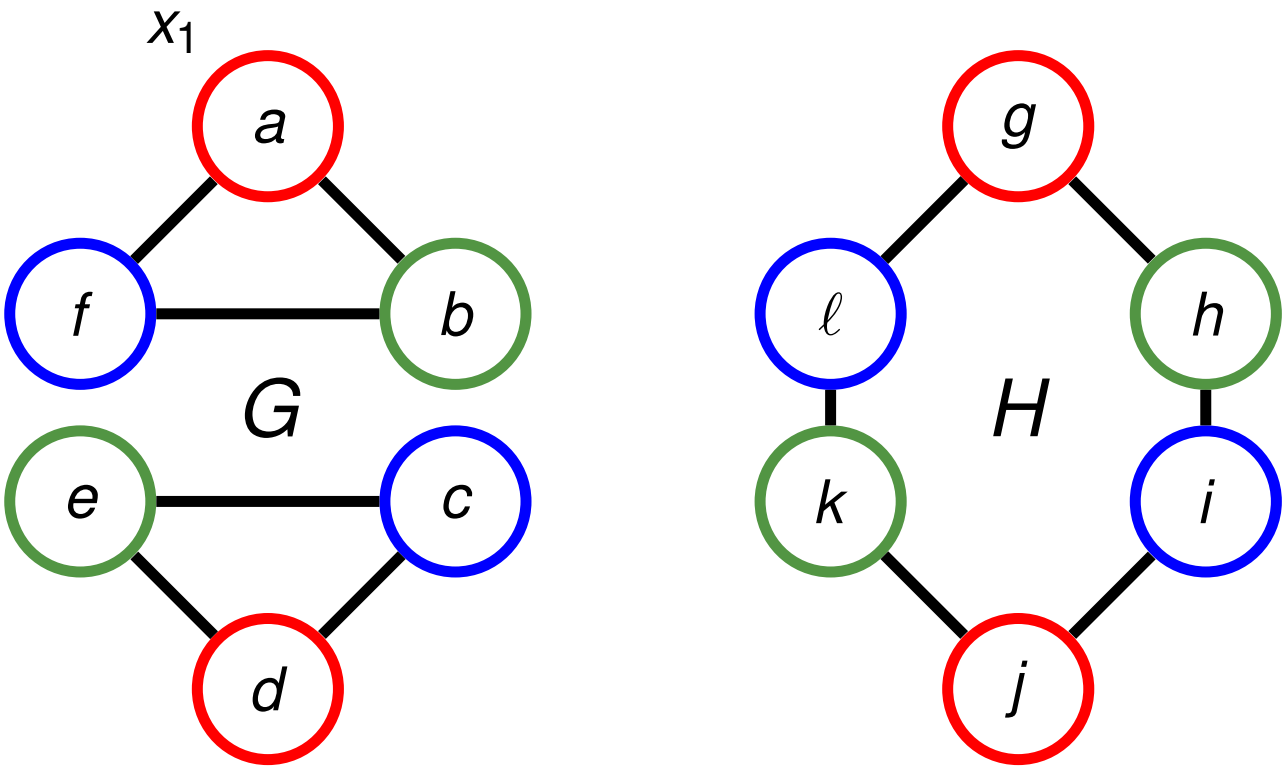

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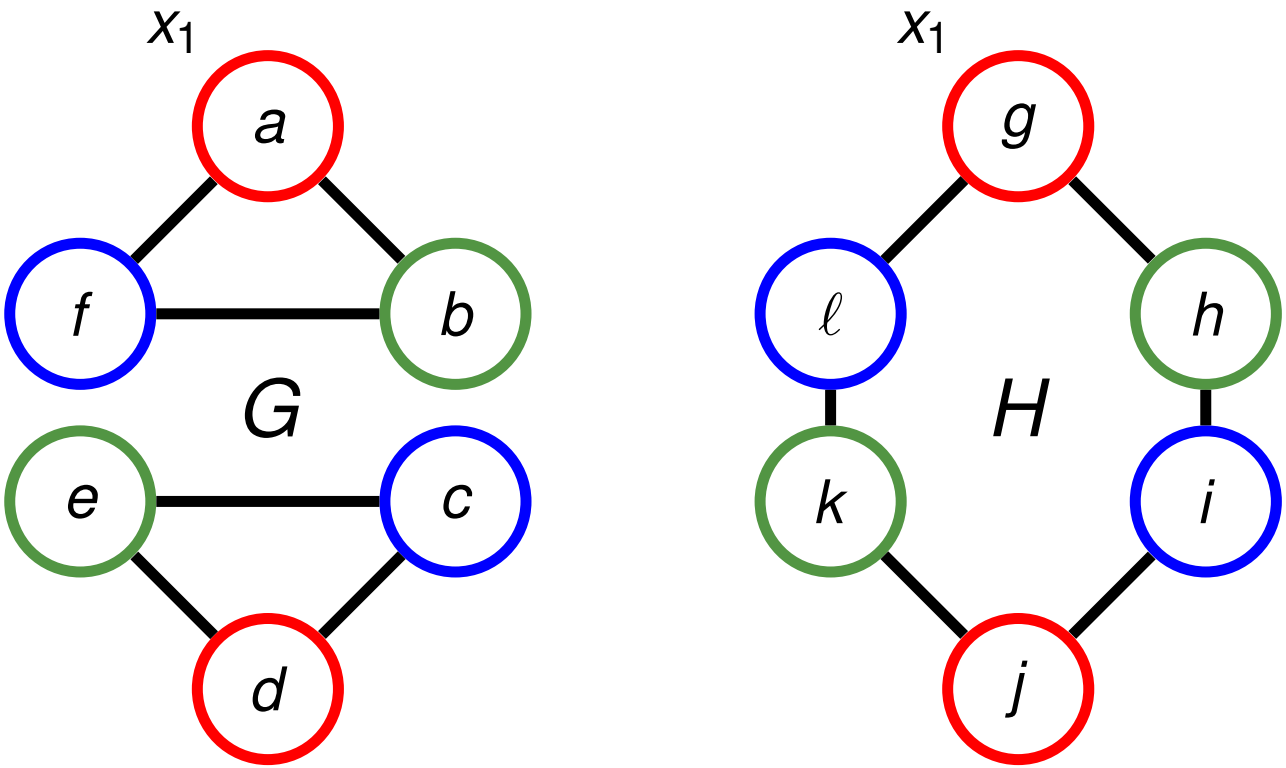

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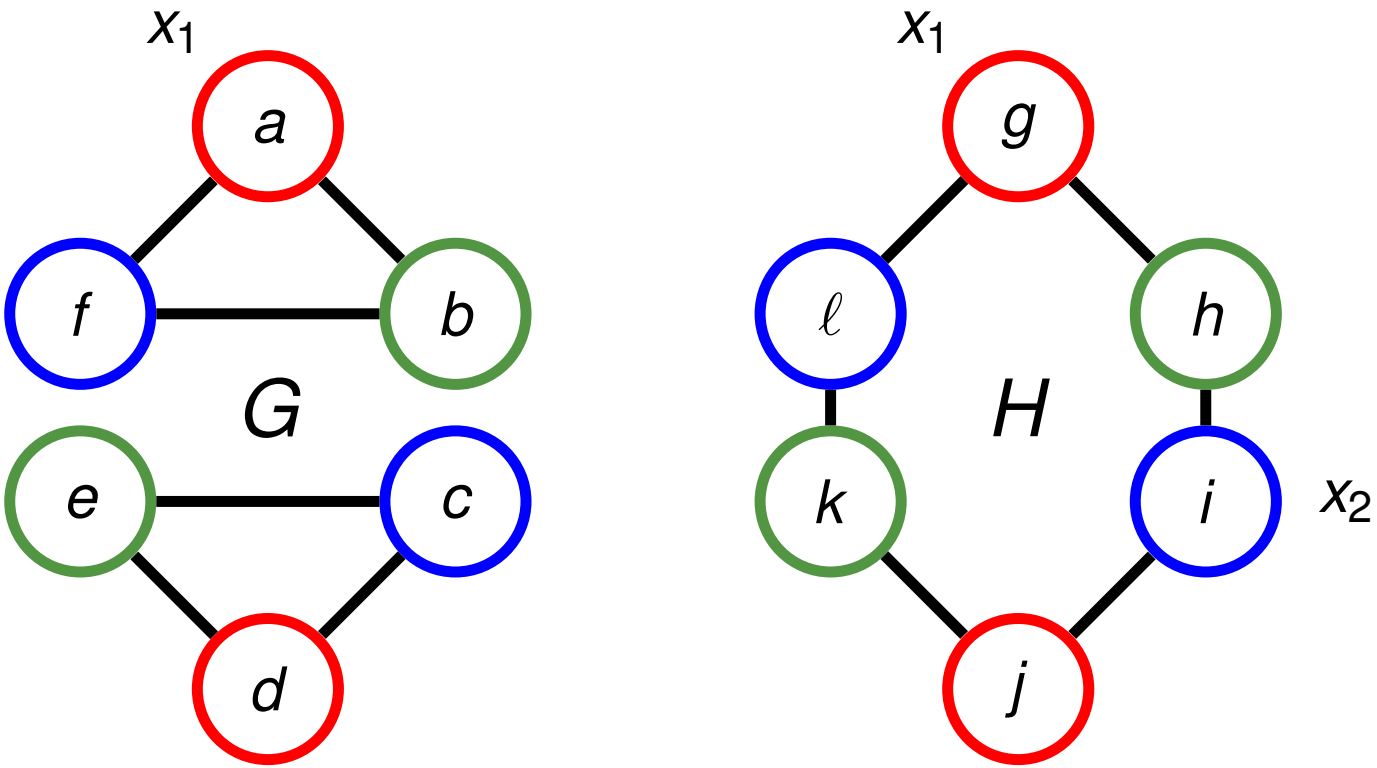

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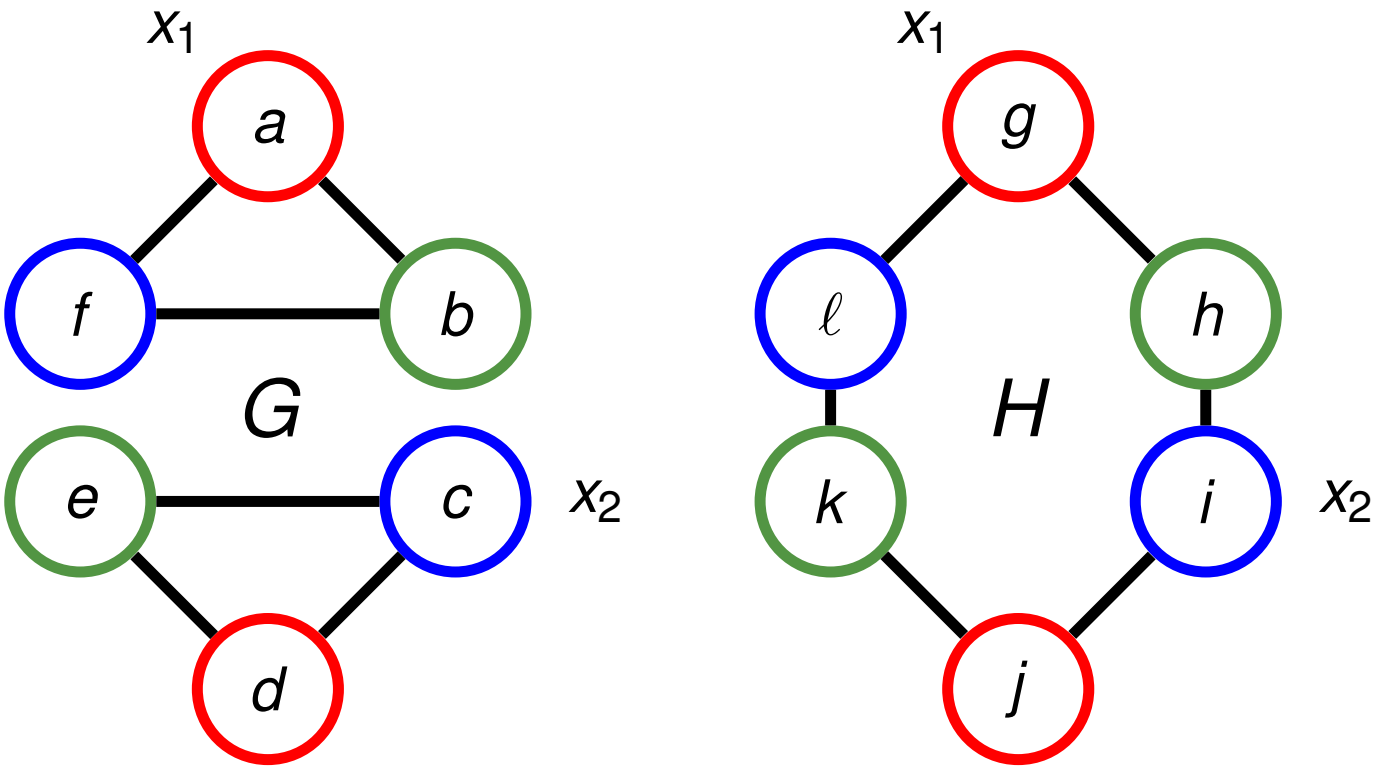

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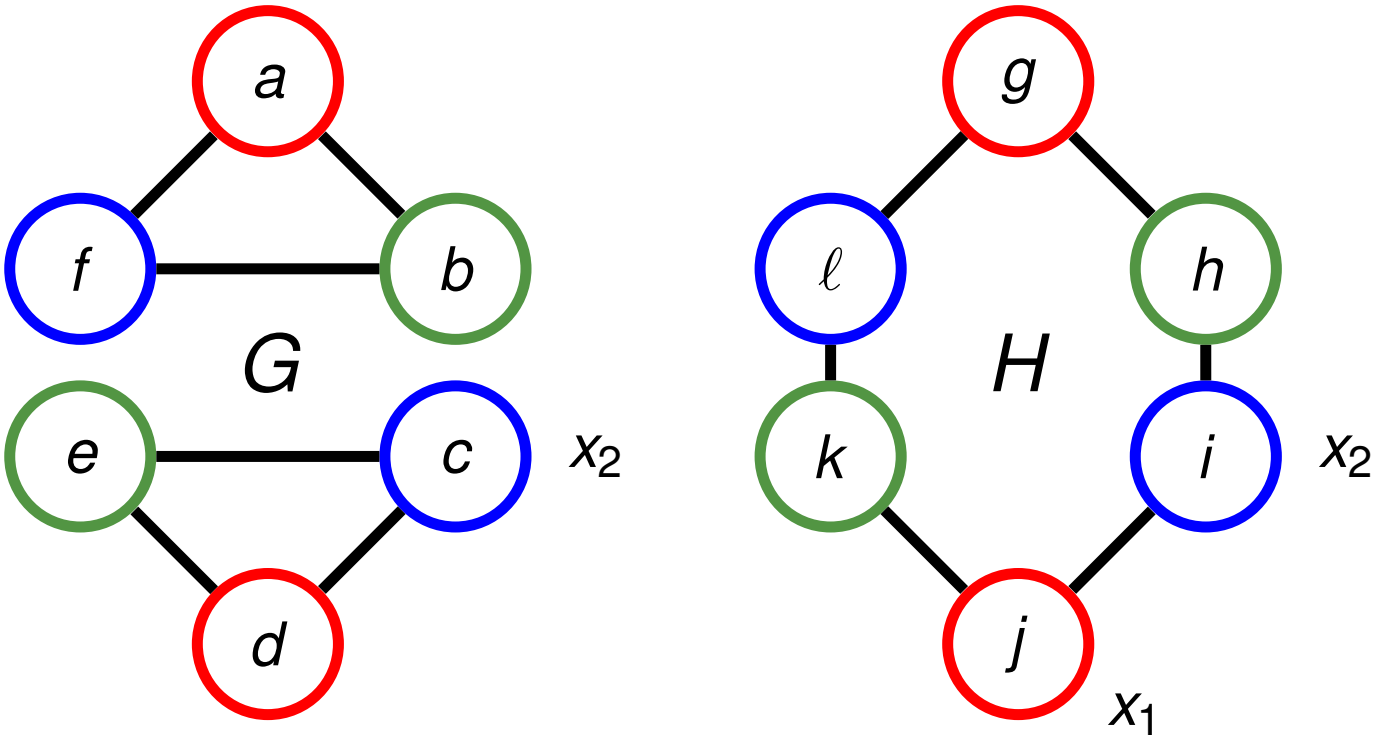

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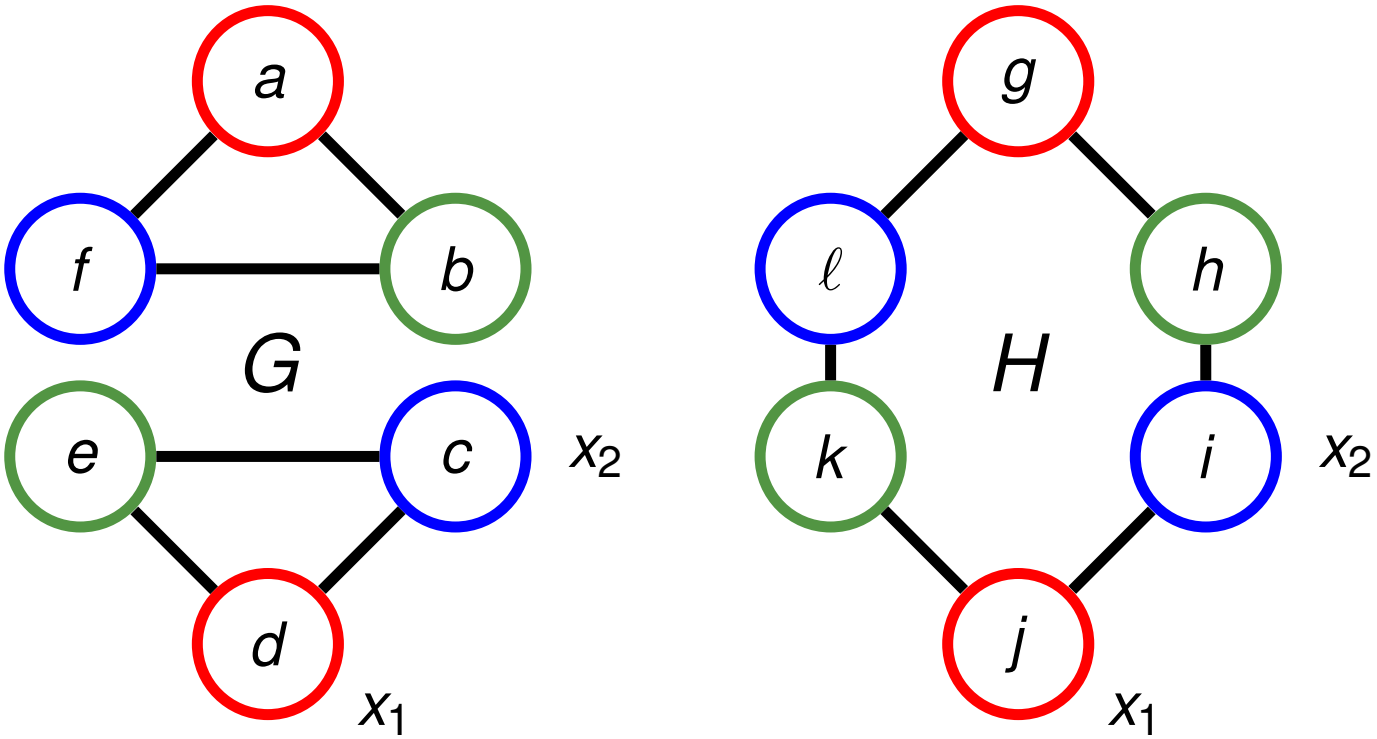

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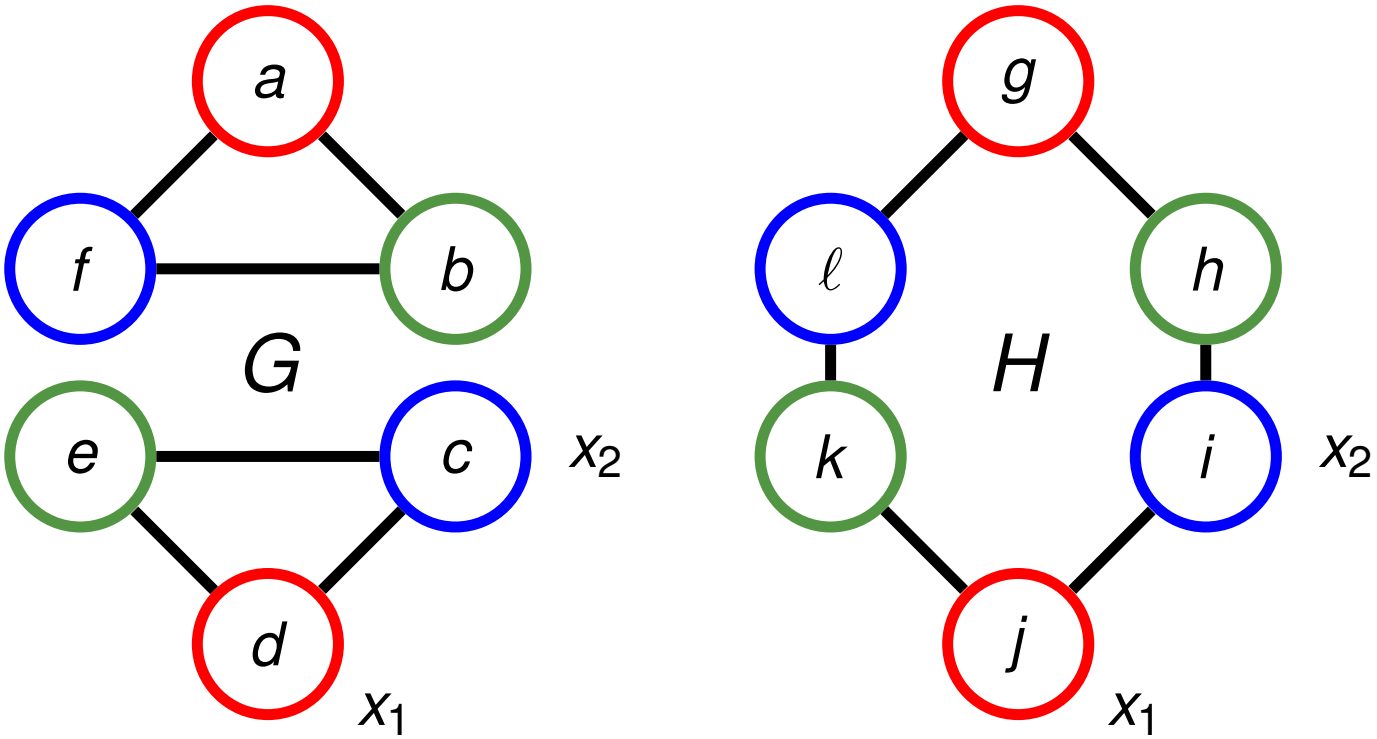

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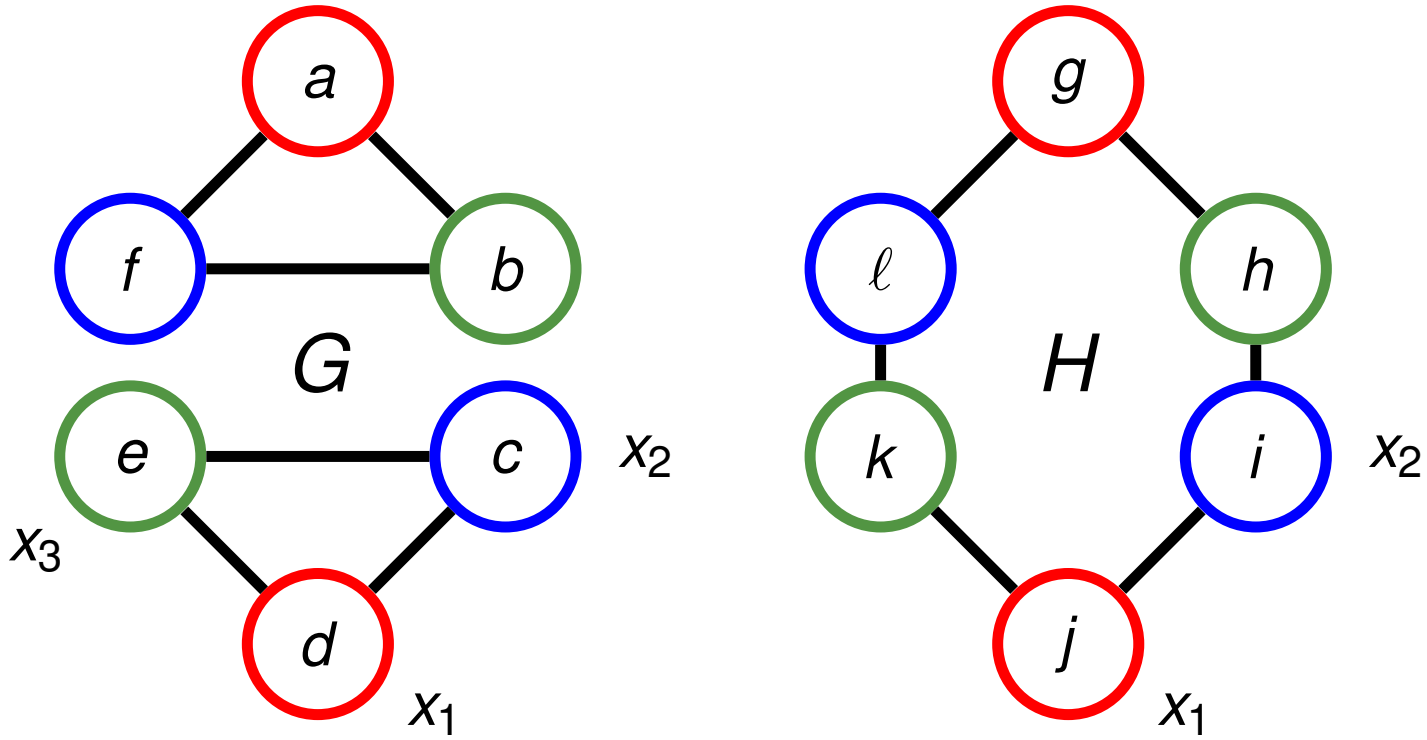

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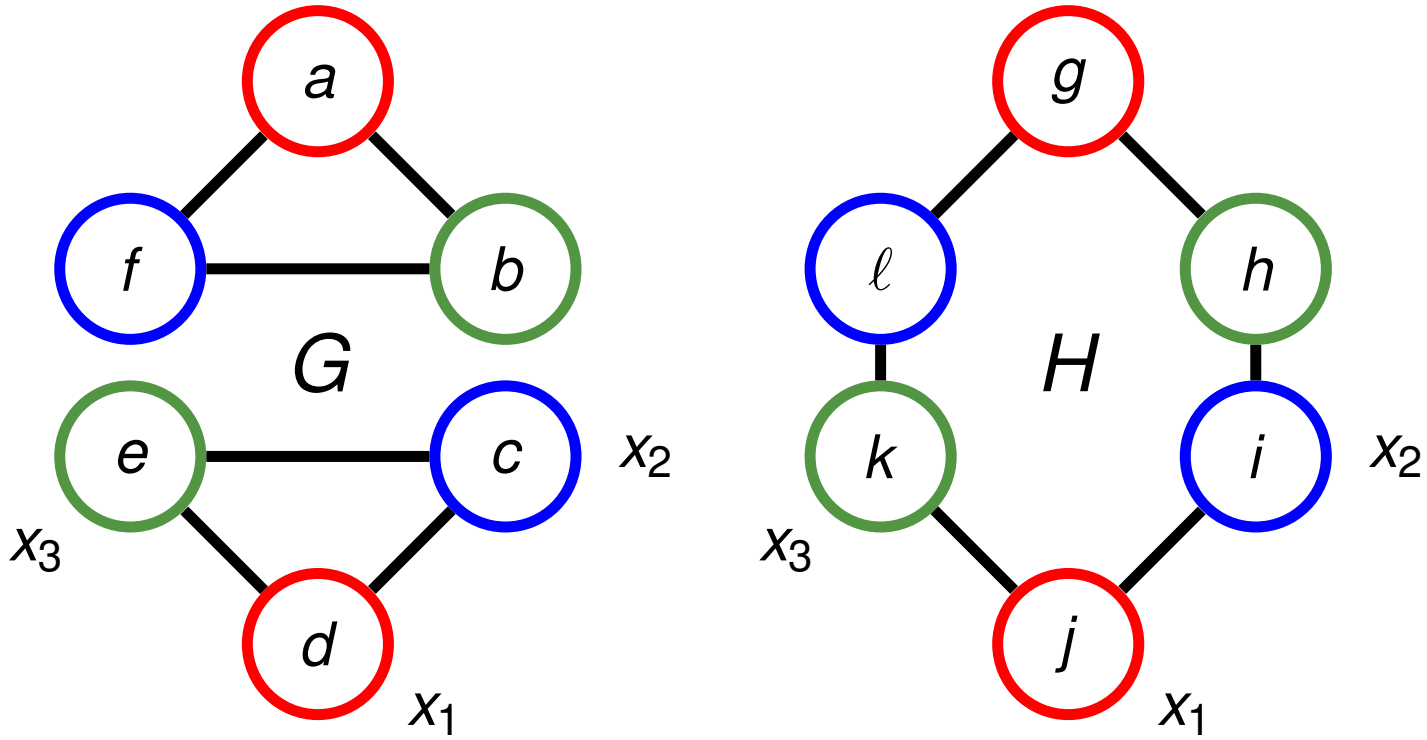

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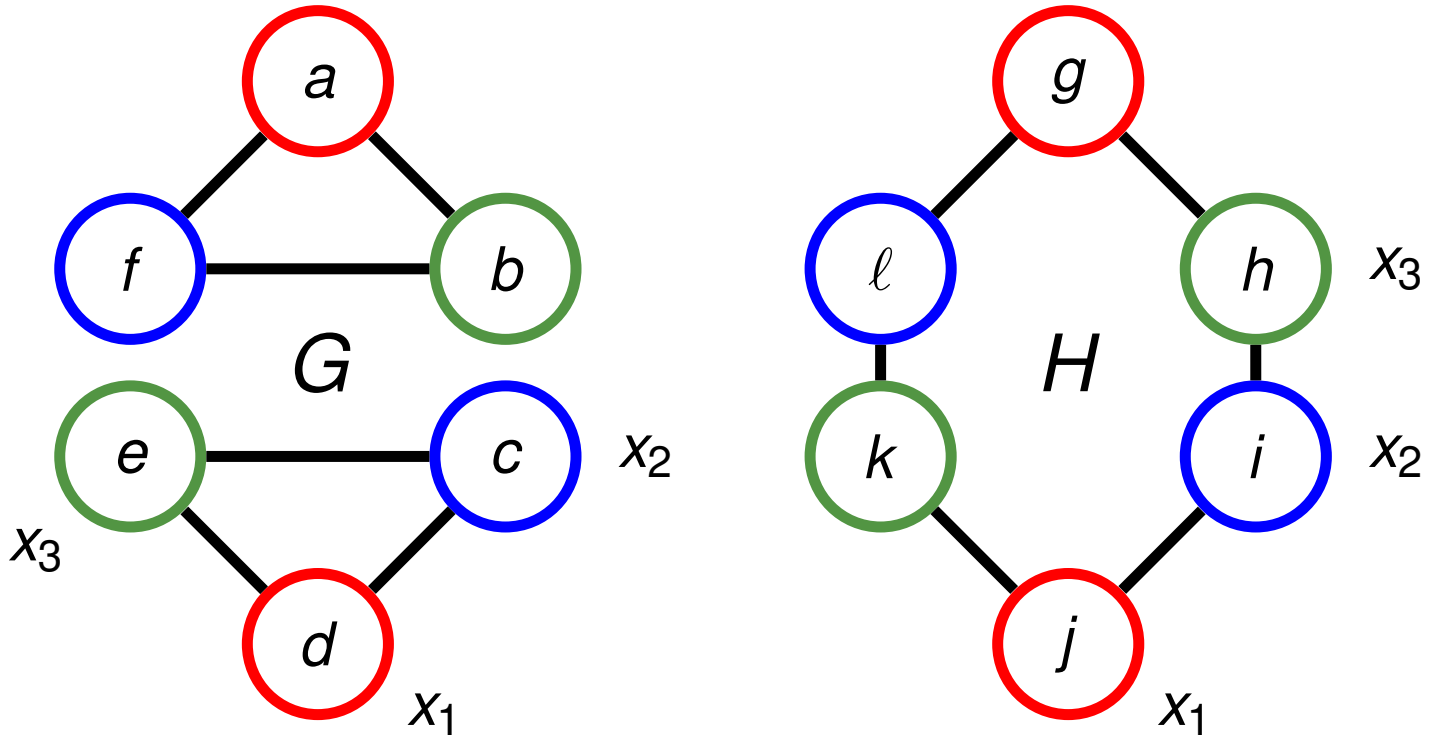

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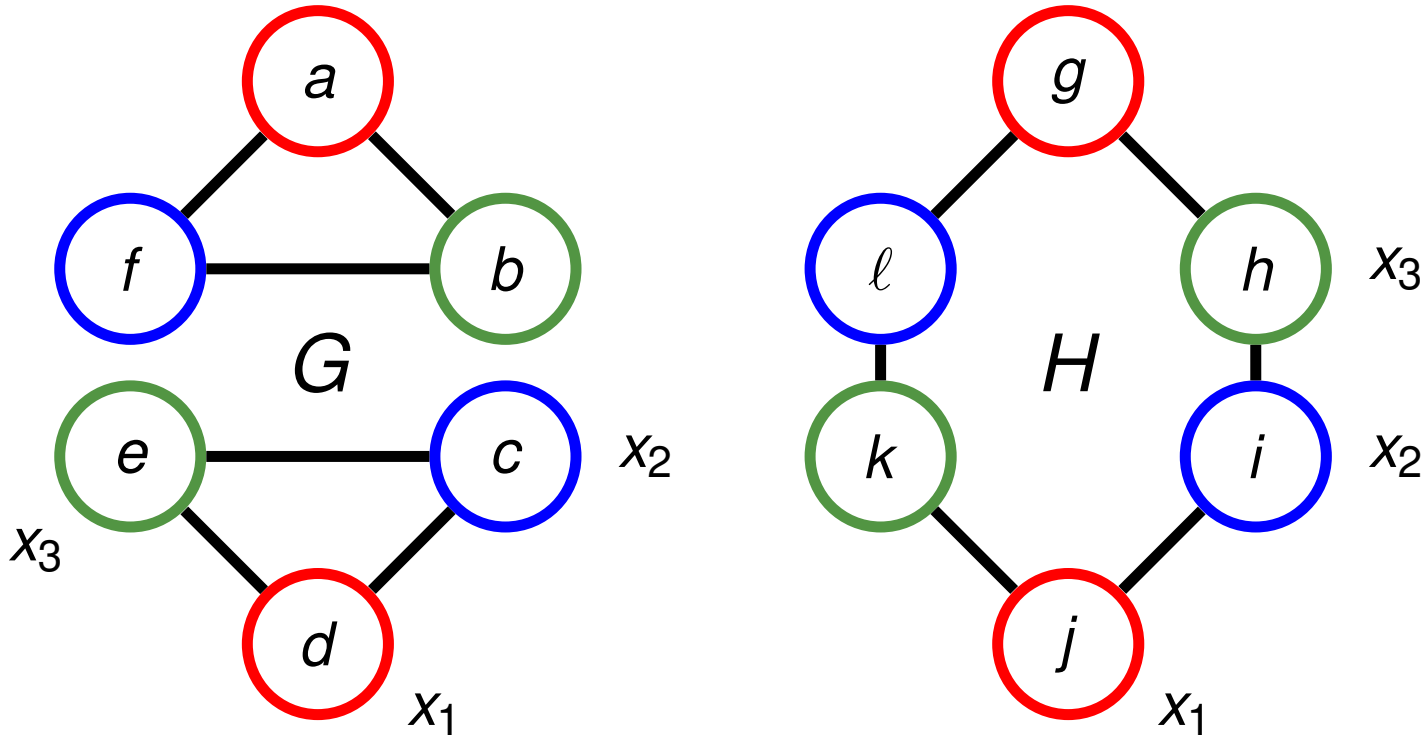

Fundamental Thm of Ehrenfeucht-Fraïssé Games

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Thm. **D** has a winning strategy on the m-move, k-pebble game on G, H iff G and H agree on all formulas using k variables and quantifier depth m.

$$G \sim_m^k H \quad \Leftrightarrow \quad G \equiv_m^k H$$

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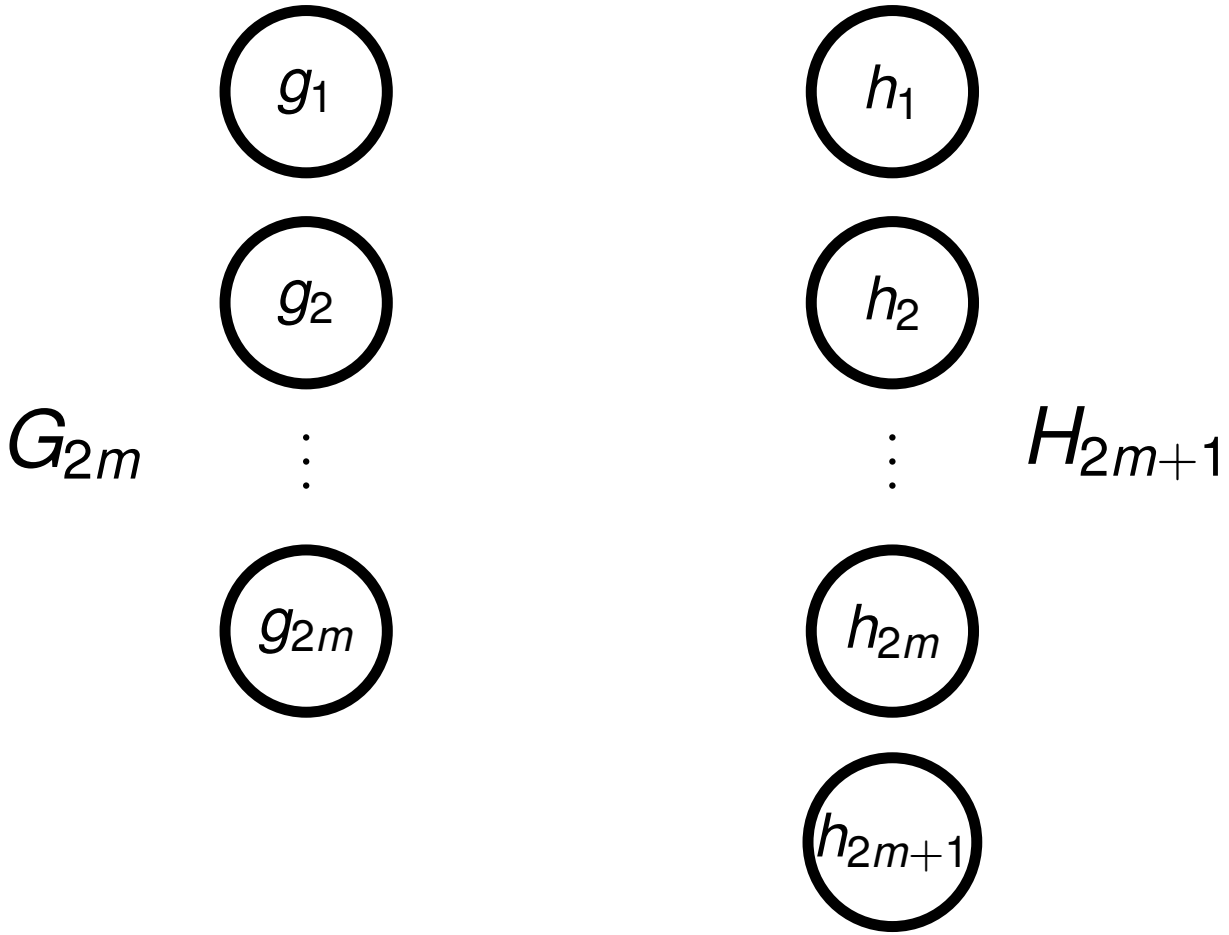

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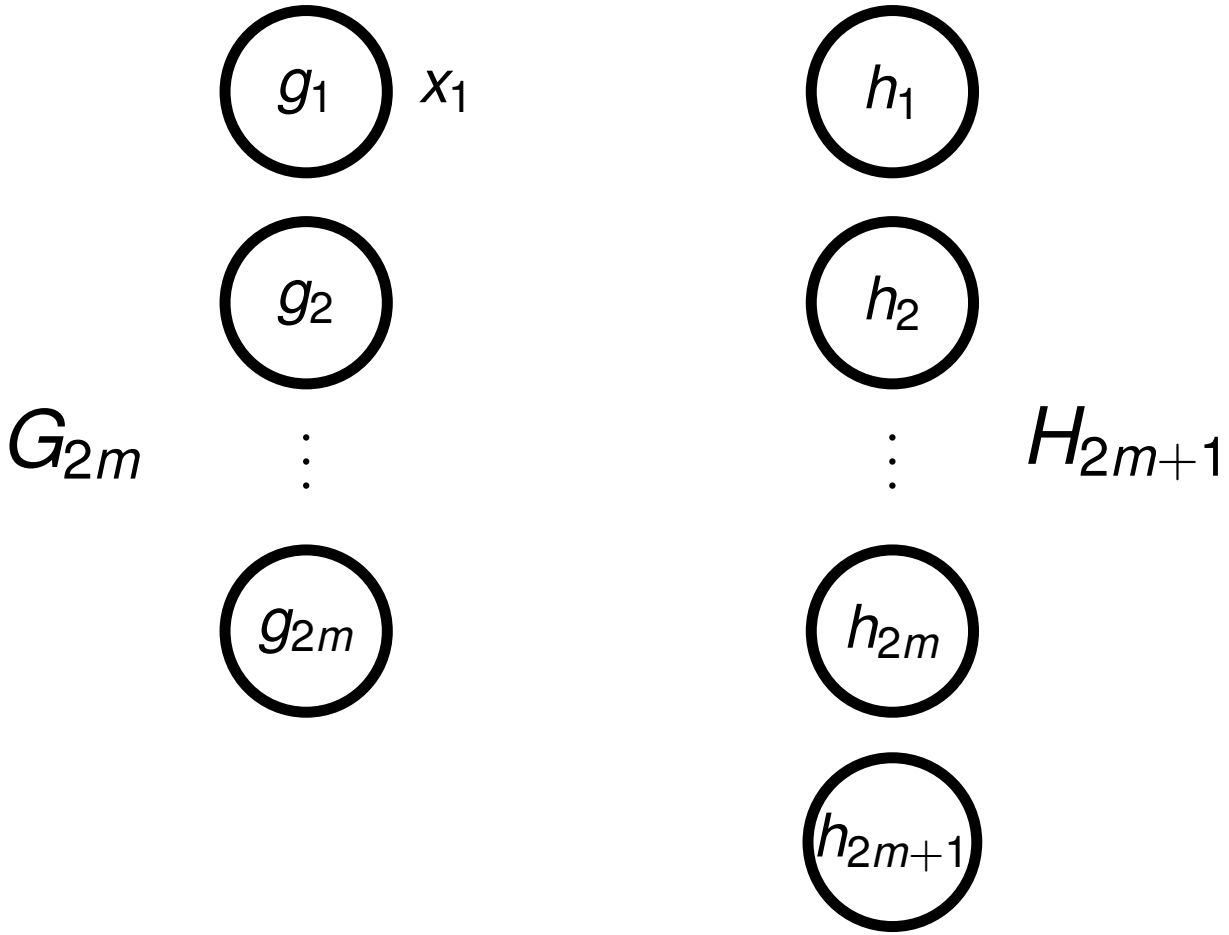

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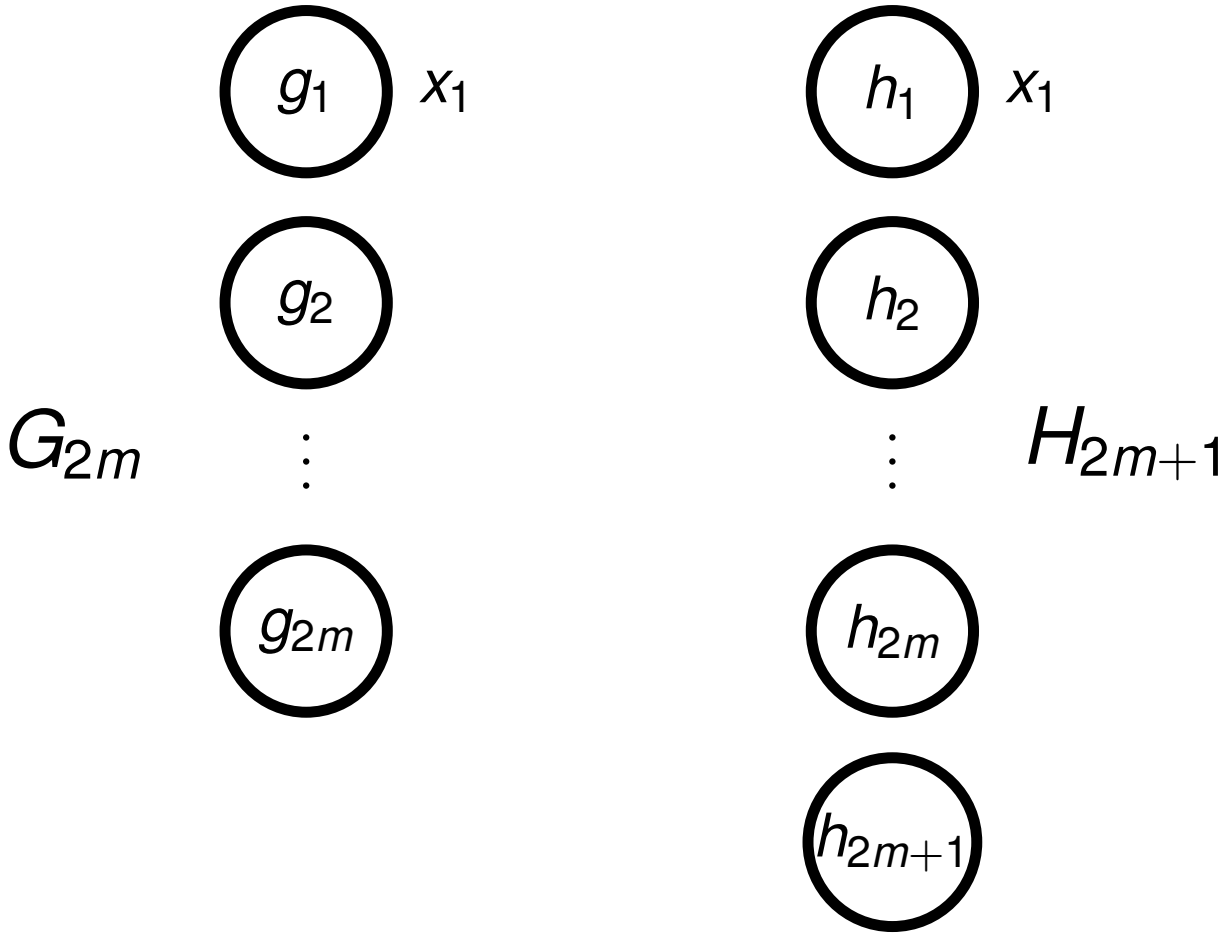

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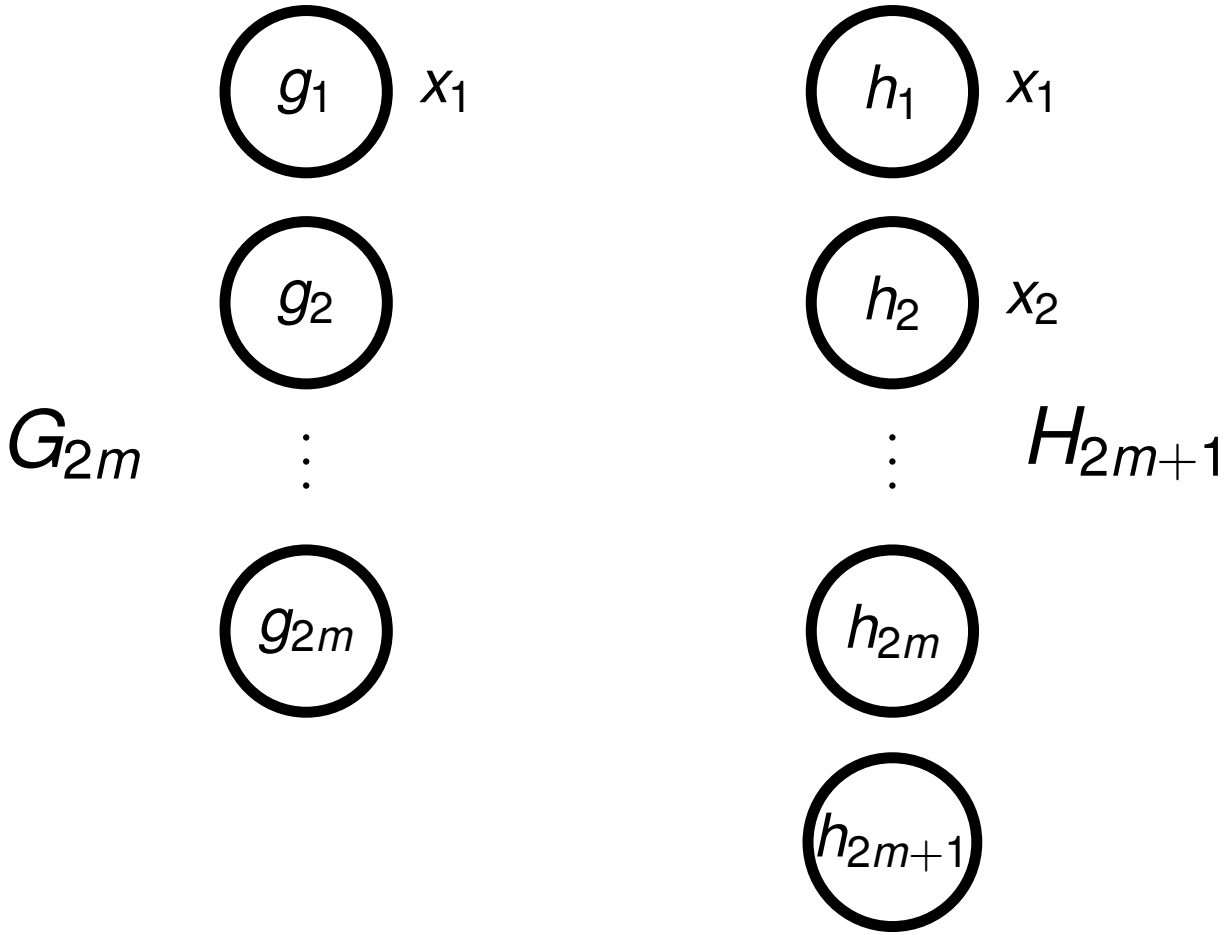

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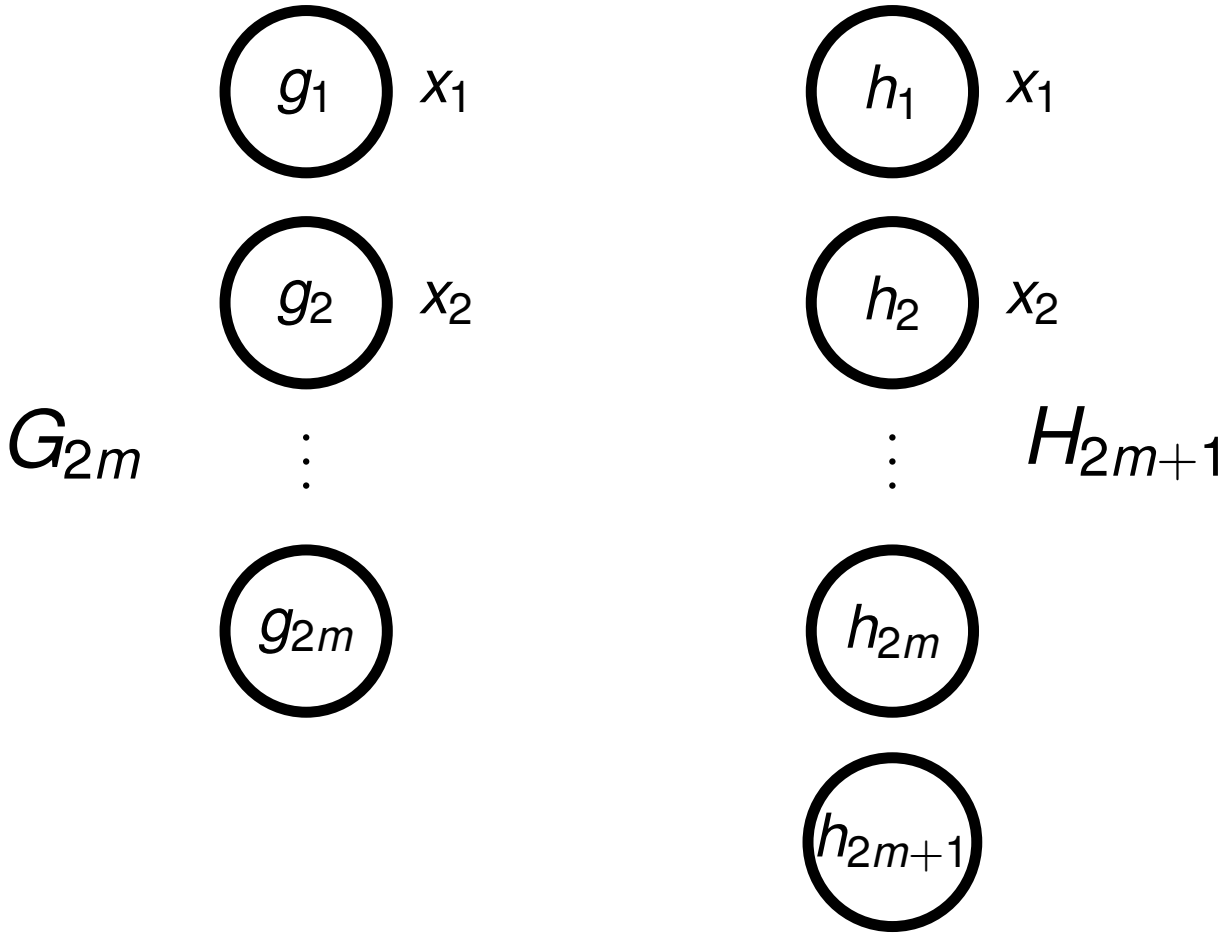

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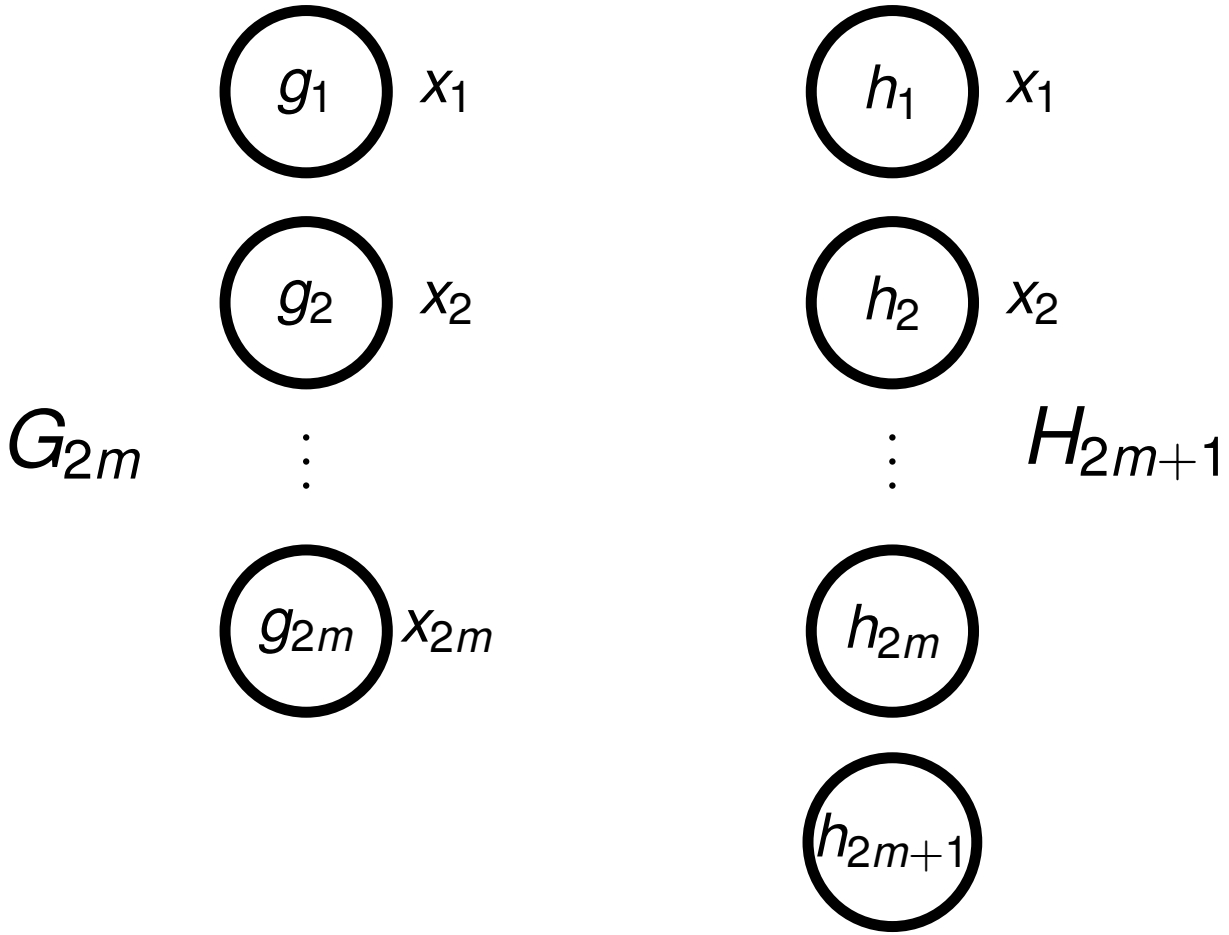

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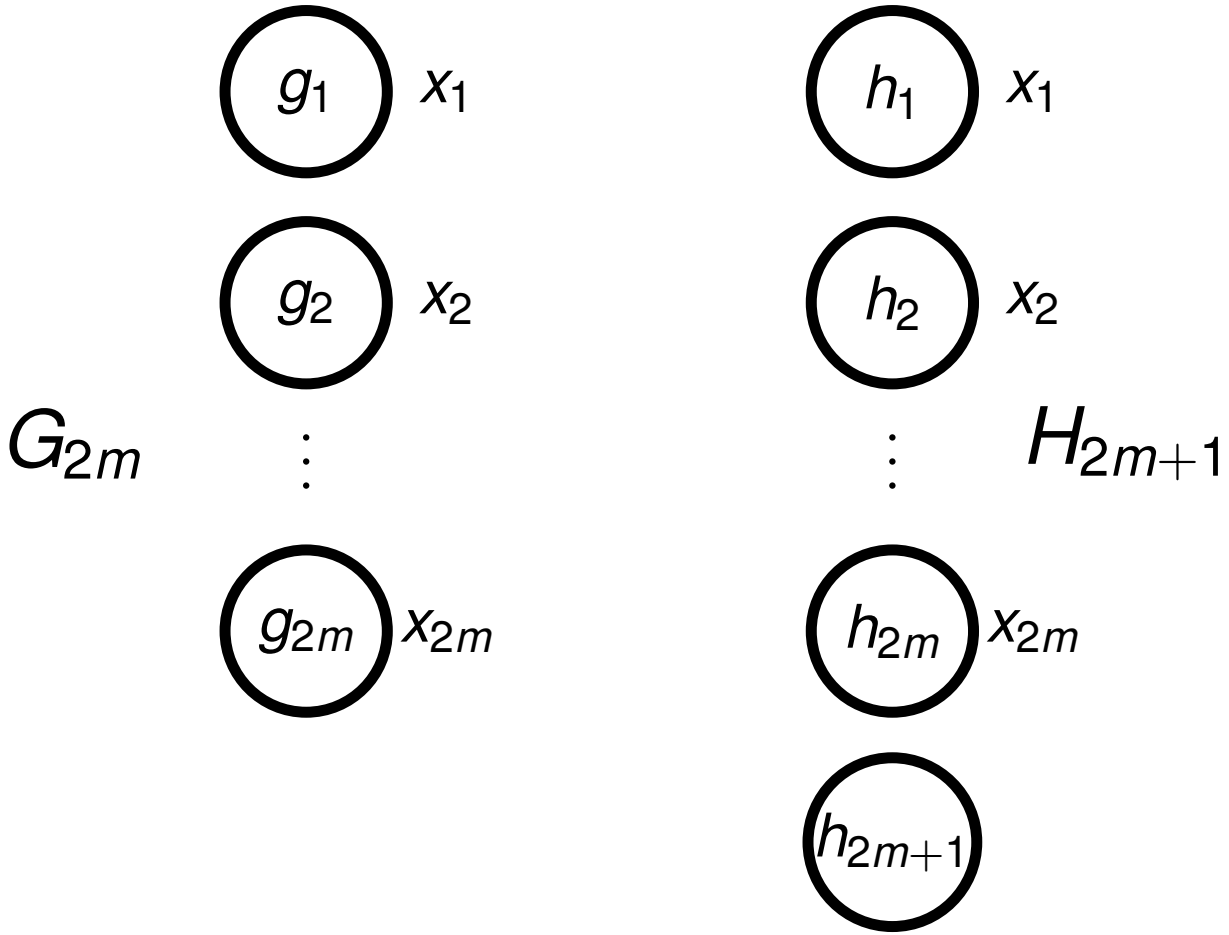

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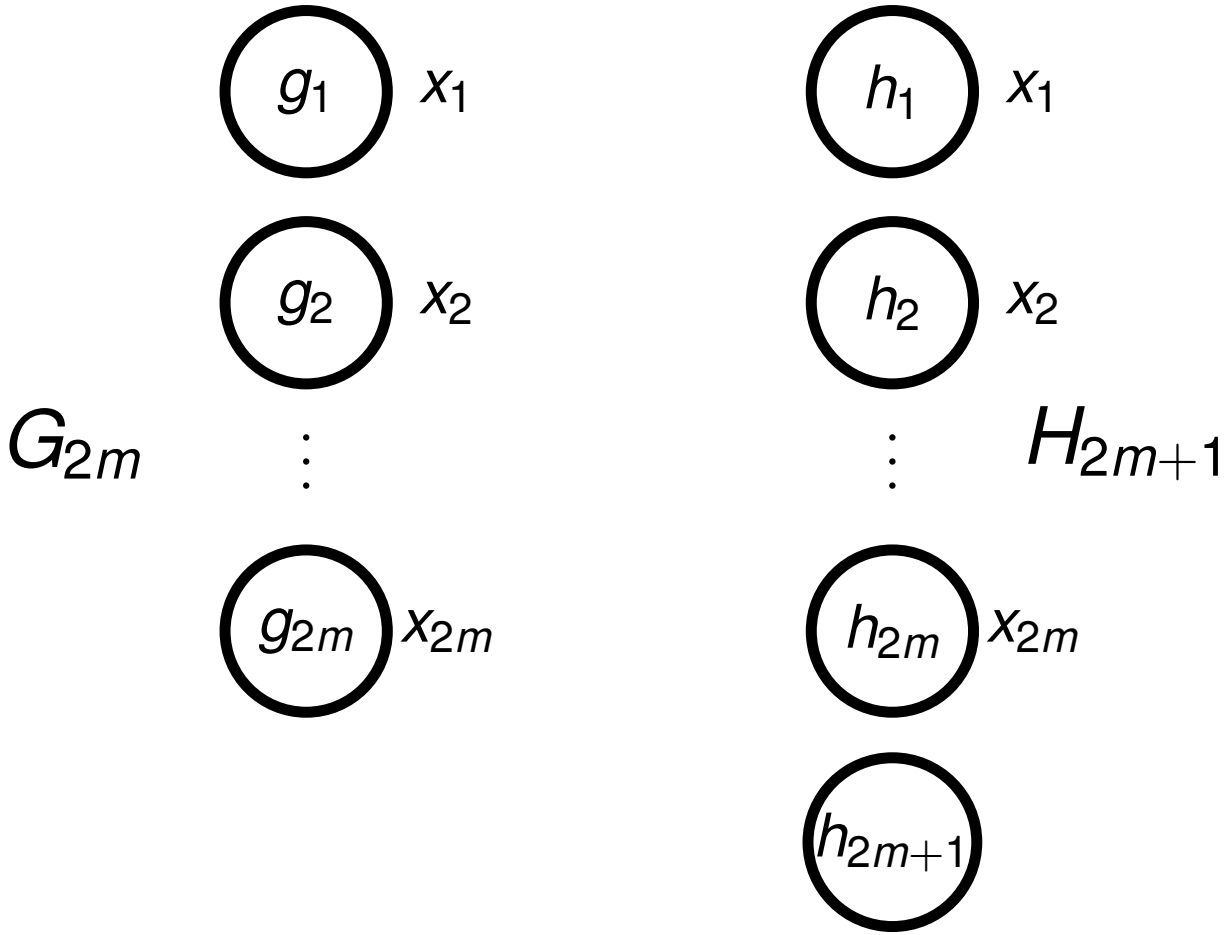

$$G_{2m} \sim^{2m} H_{2m+1}$$

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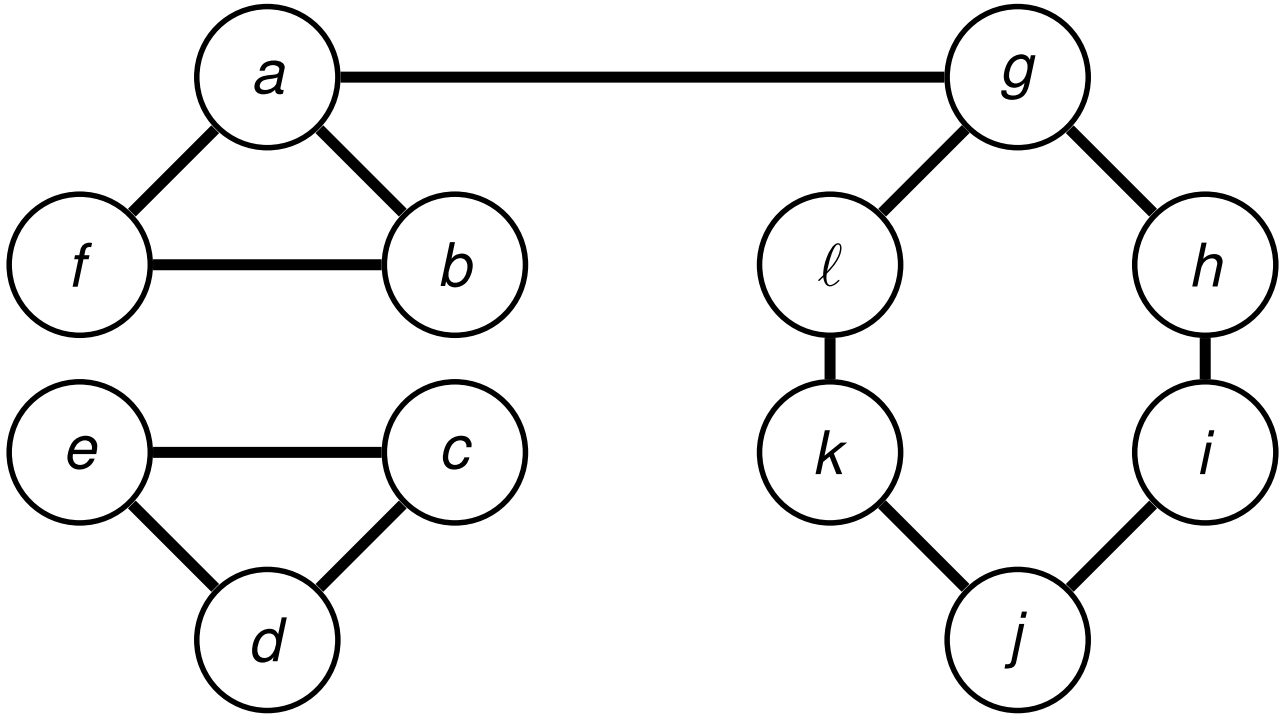
Start with a colored graph, and repeatedly color each vertex by how many neighbors it has of each color.



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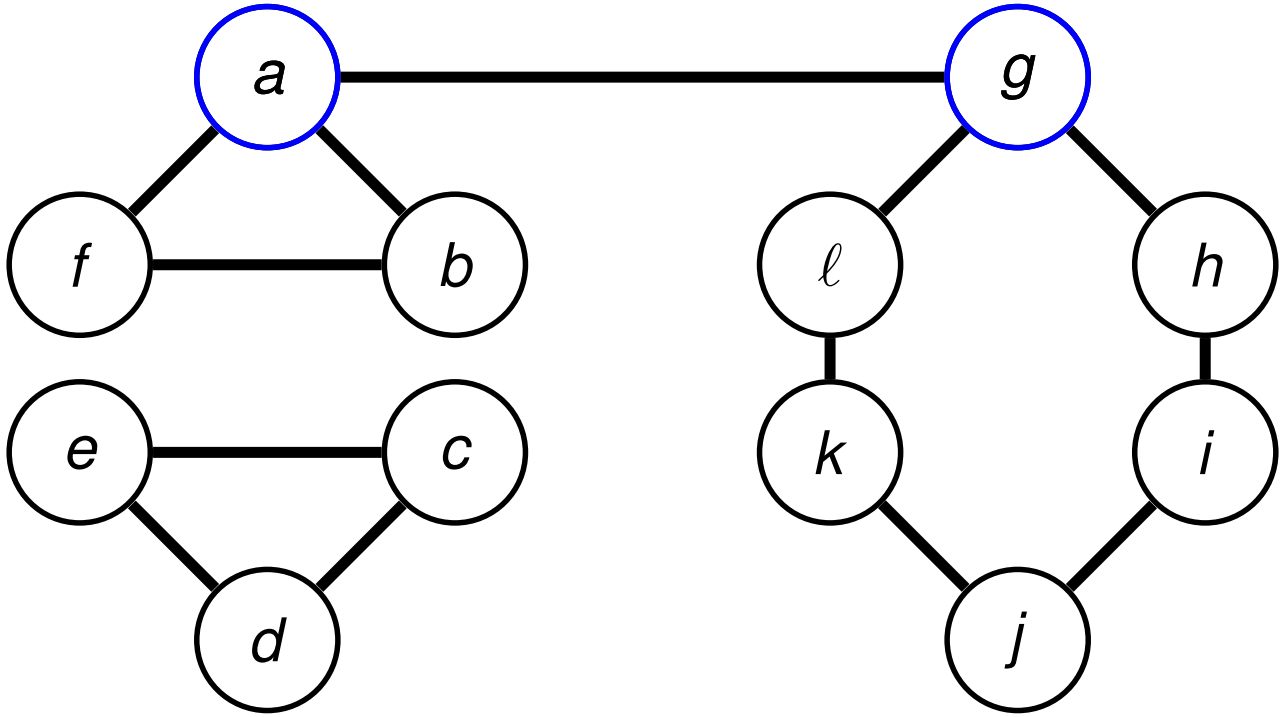

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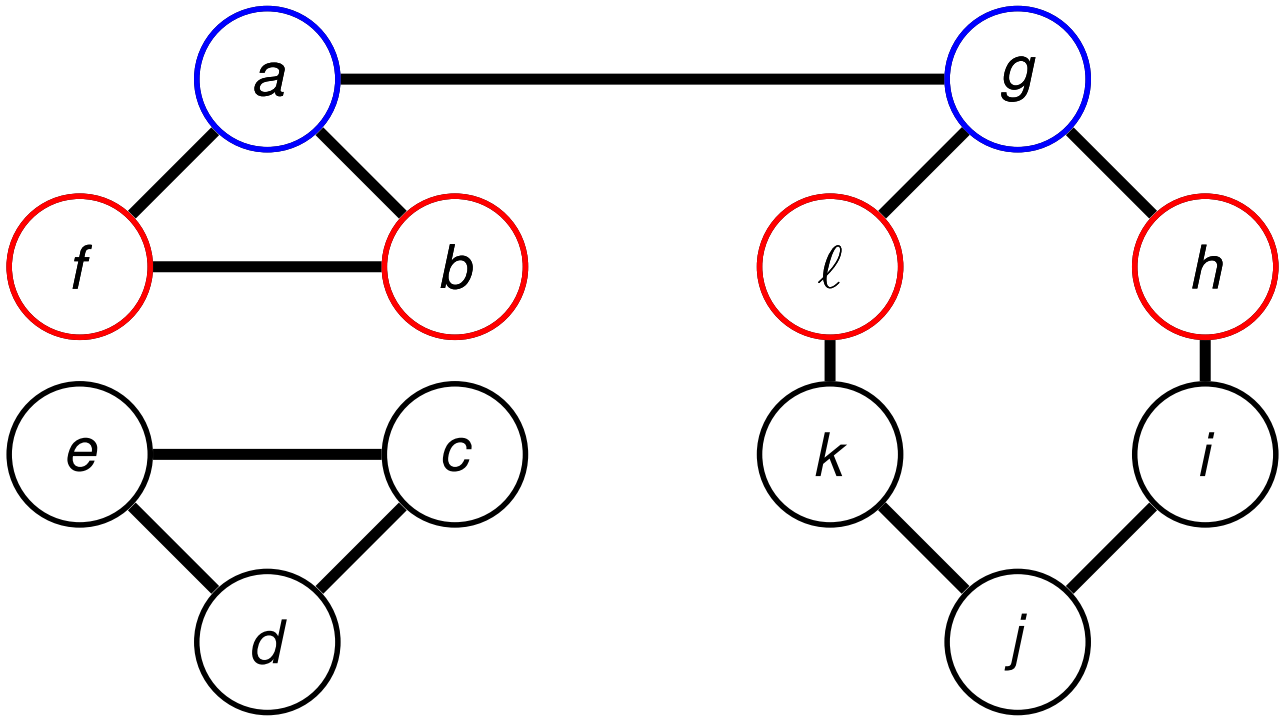

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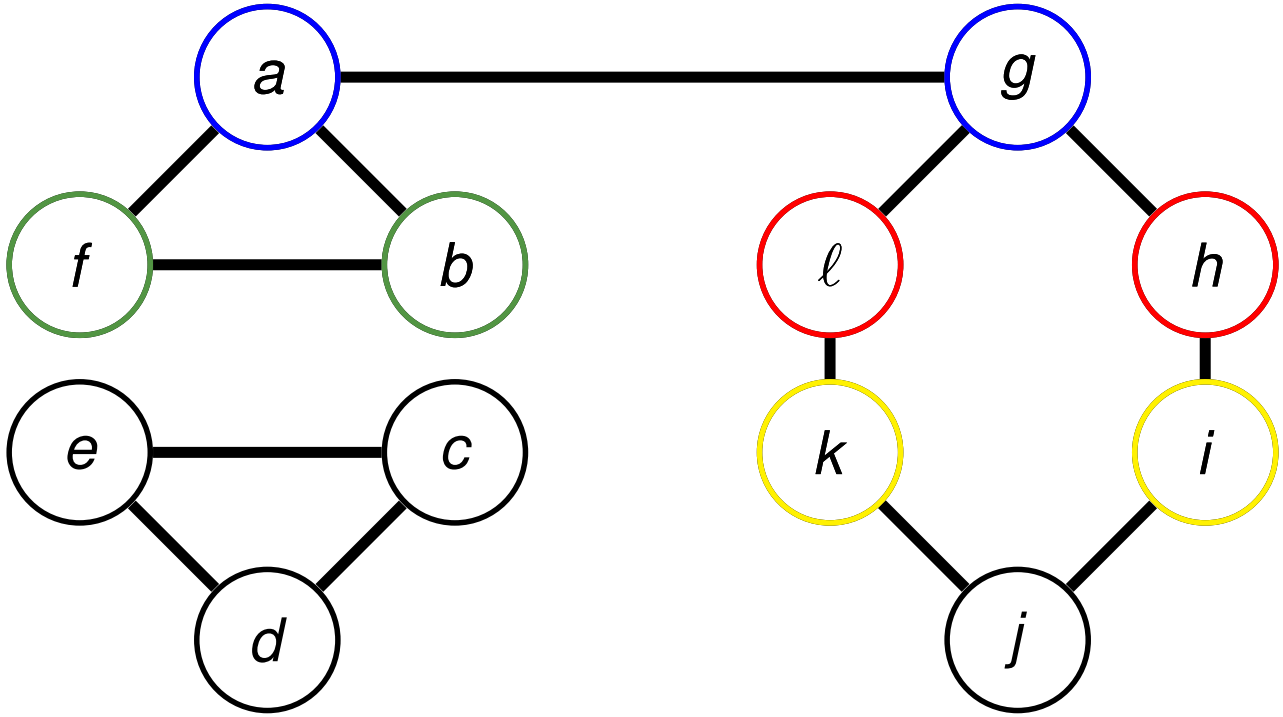

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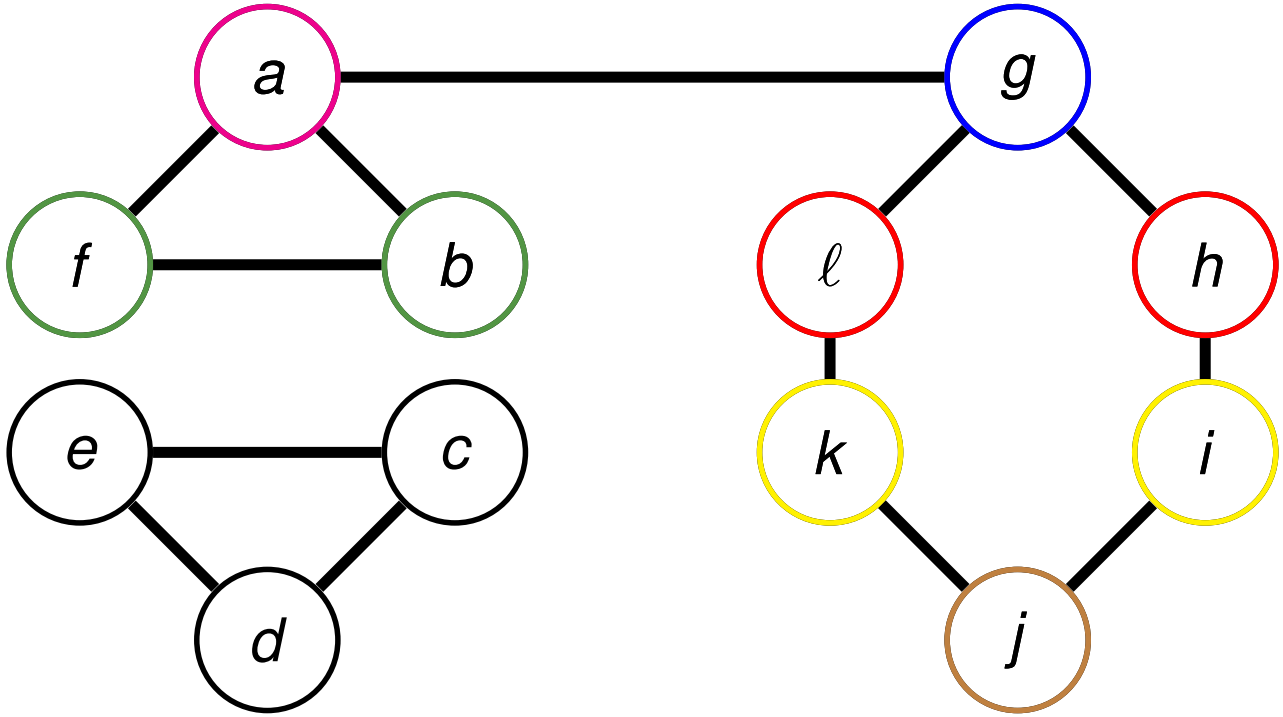

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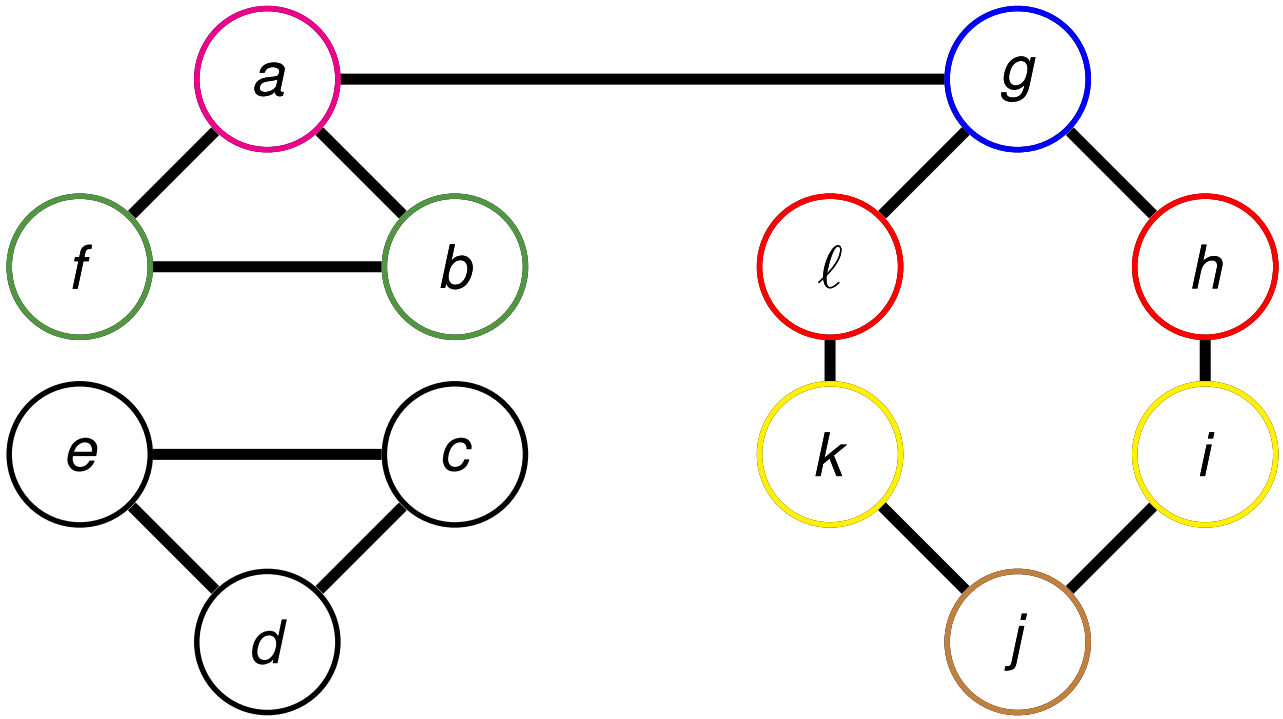

Thm. Stable Coloring of Vertices $=$ C^2 type.

Round m of stable coloring is quantifier depth of C^2 formula.

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Thm. [Babai, 2015] GI $\in \mathrm{DTIME}[n^{\log^7 n}]$. (Before this it was only known that GI $\in \mathrm{DTIME}[n^{\sqrt{n}}]$.)

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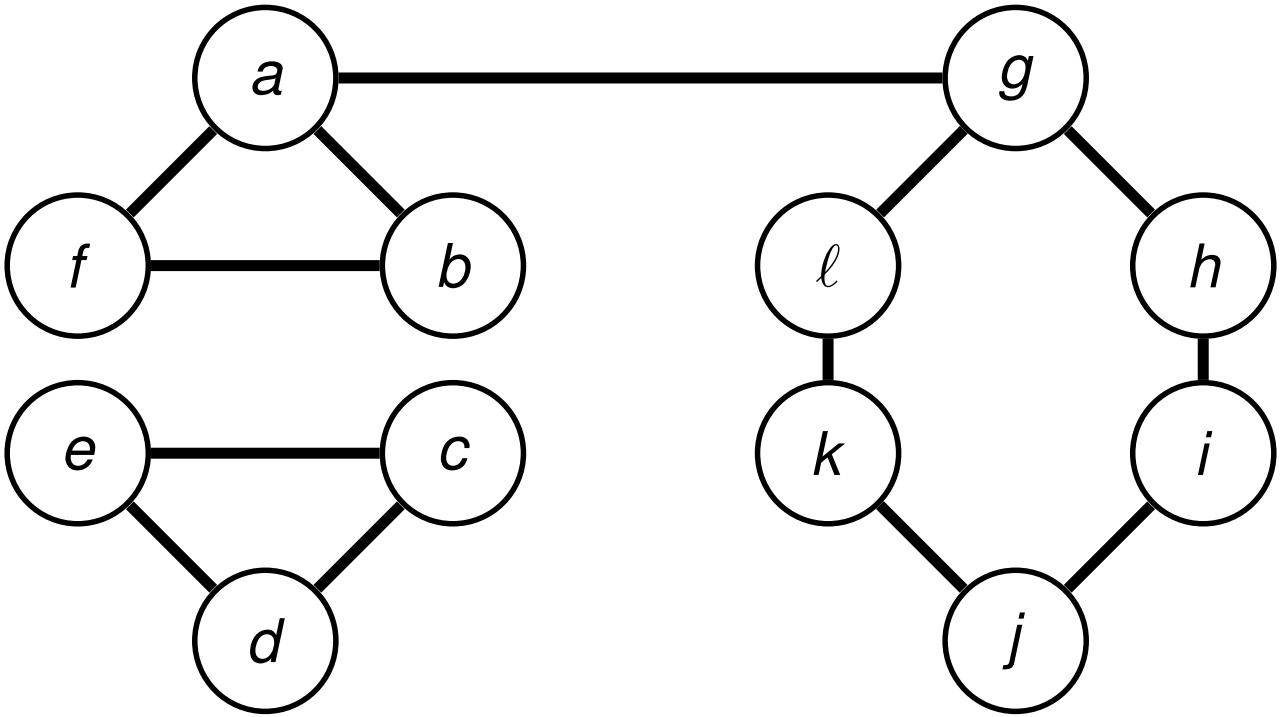
proof: Apply arbitrary $\mathrm{FO(LFP)}$ formula to the canonical form of the input graph. □

Particularizing Means Uniquely Coloring Some Vertex



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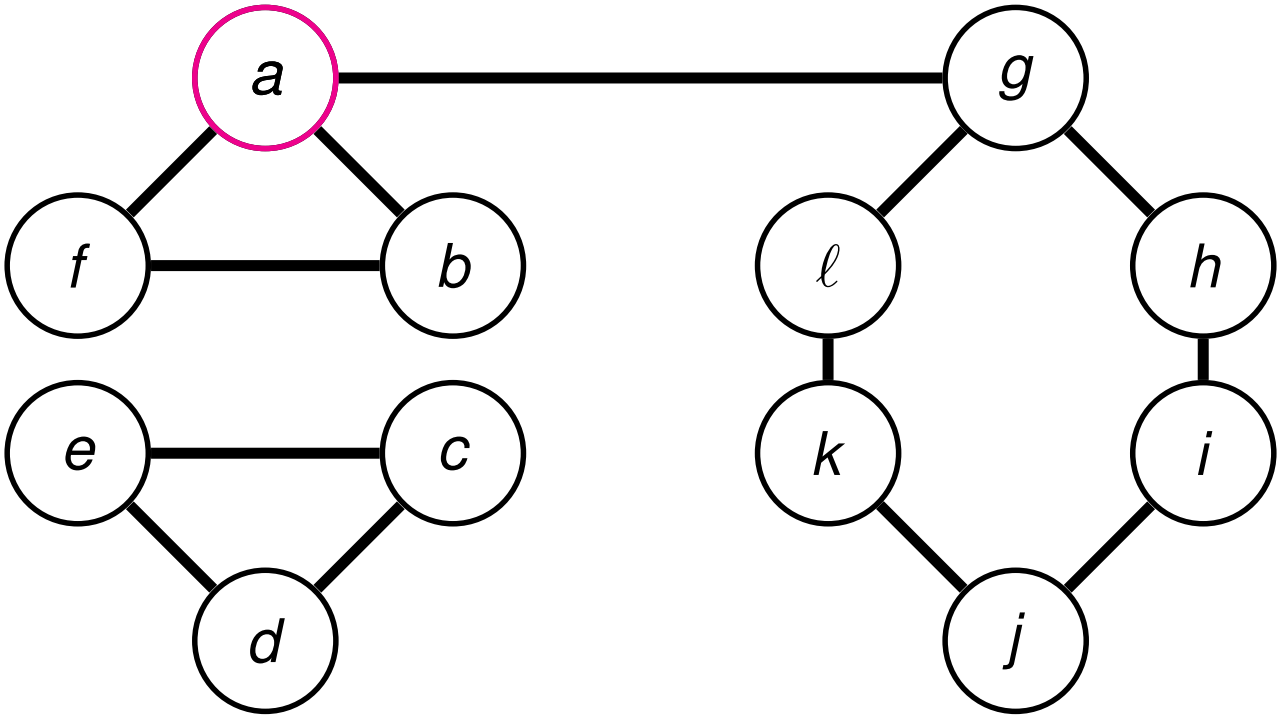

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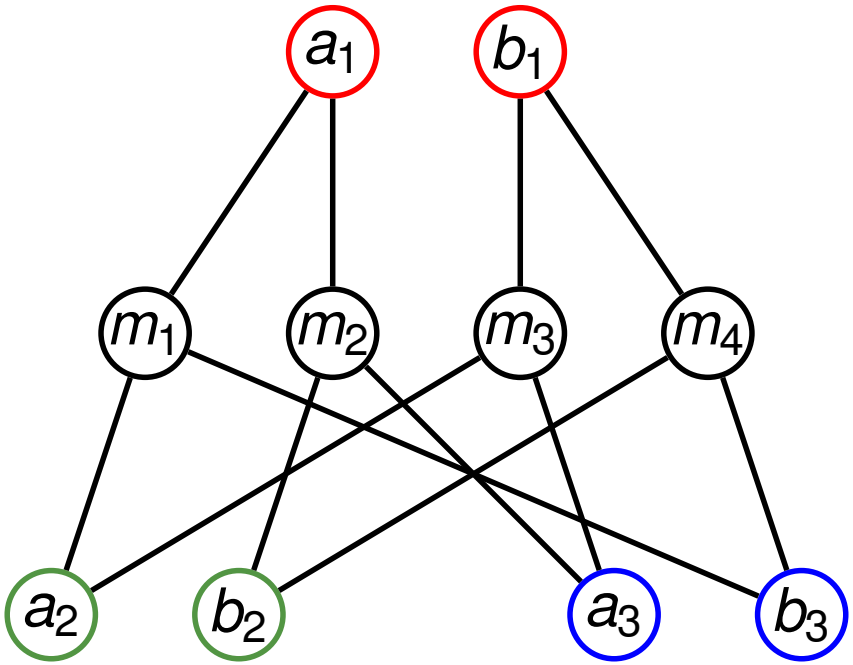
Thm. [CFI] No!

A simple graph property (now called the CFI property) checkable in $\mathrm{DTIME}[n]$, requires $v = \Omega(n)$ variables to express in C^v. Thus, CFI $\in \mathrm{OIP} - \mathrm{FPC}$

Proof of CFI Thm



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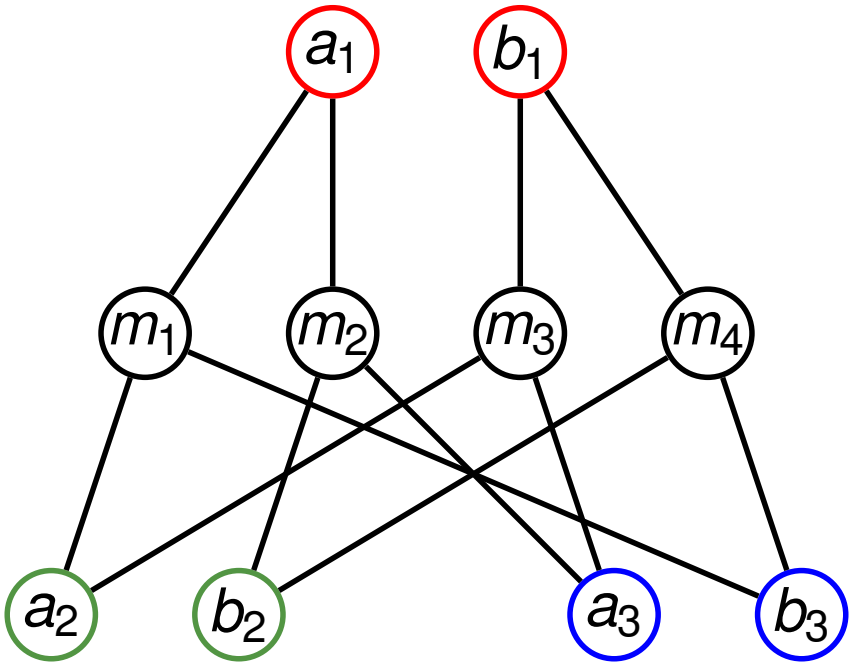
CFI Gadget X: Each m_i adjacent to an even number of a_j's.



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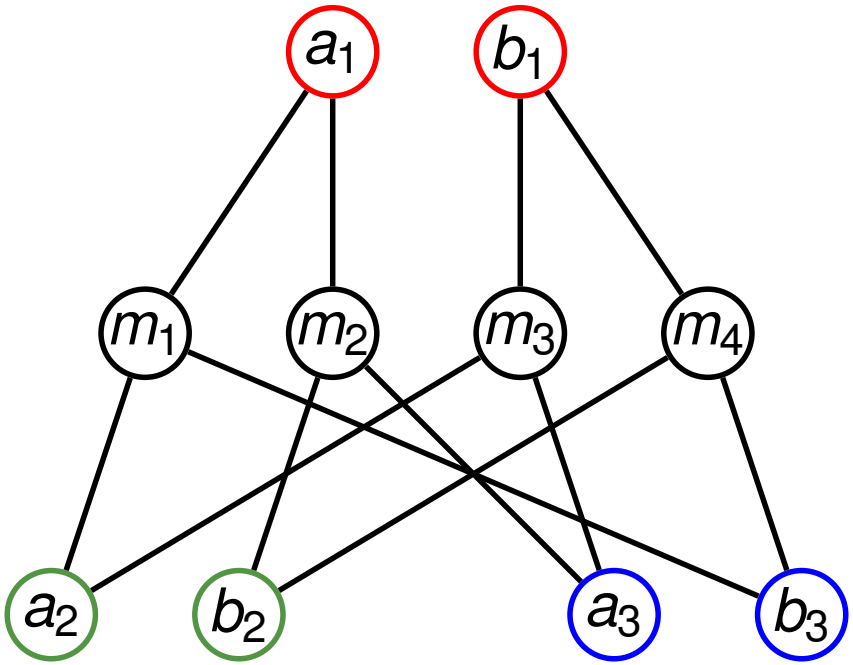
Automorphisms of X: switch an **even number** of $(a_i b_i)$ pairs.



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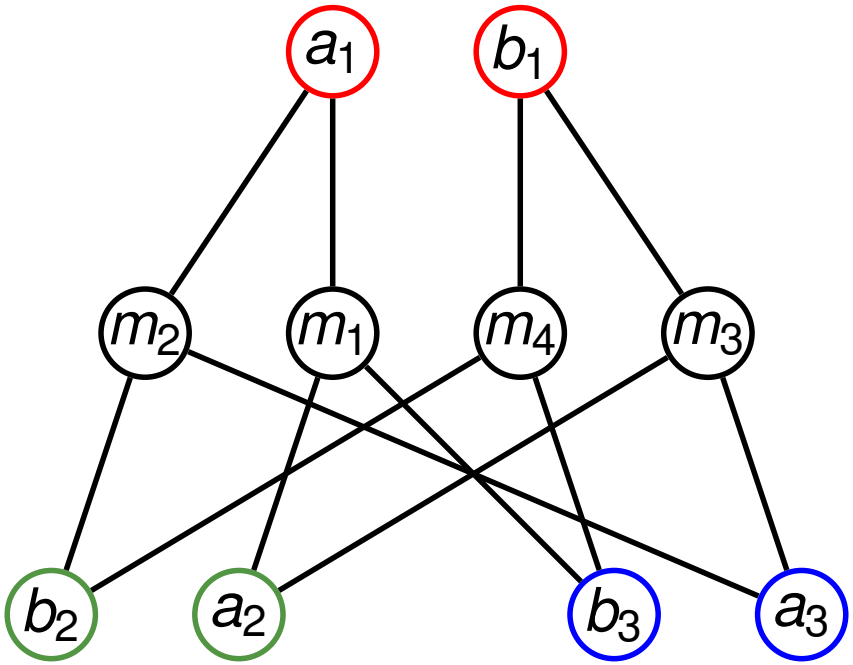

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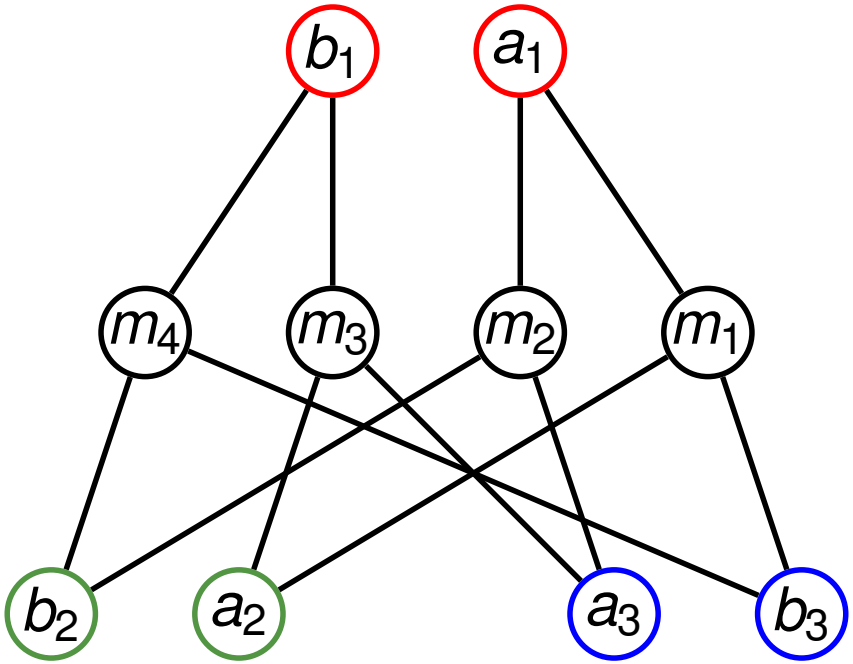

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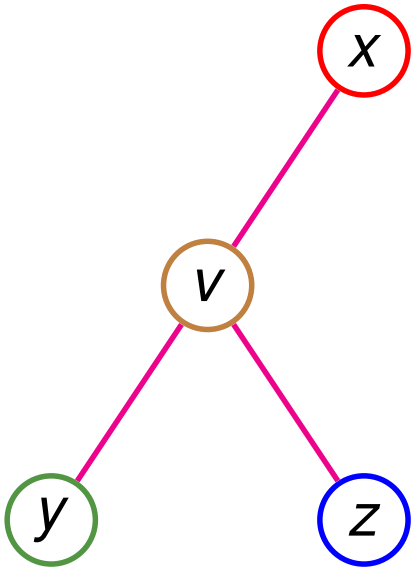
- Let $X(G_n)$ be the result of replacing each vertex $v \in V^{G_n}$ by a copy of X of v's color.
- Thus $X(G_n)$ has color class size 4.



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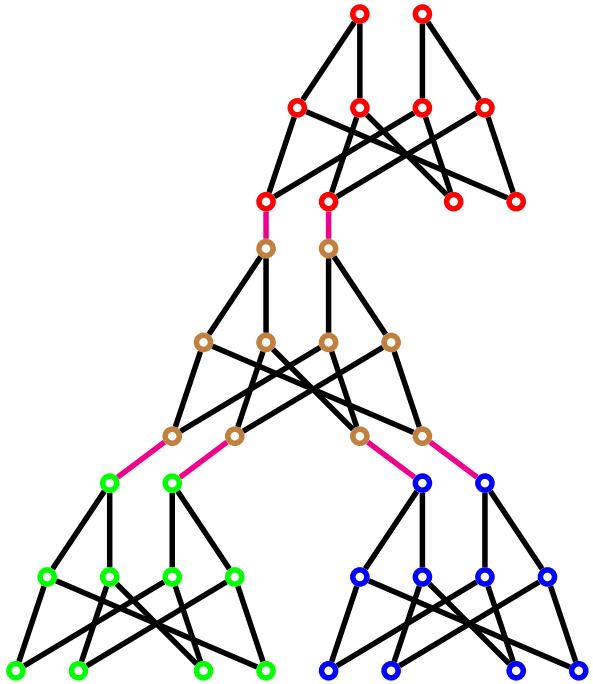

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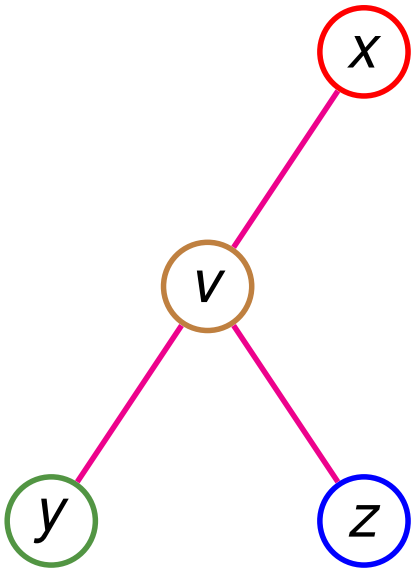

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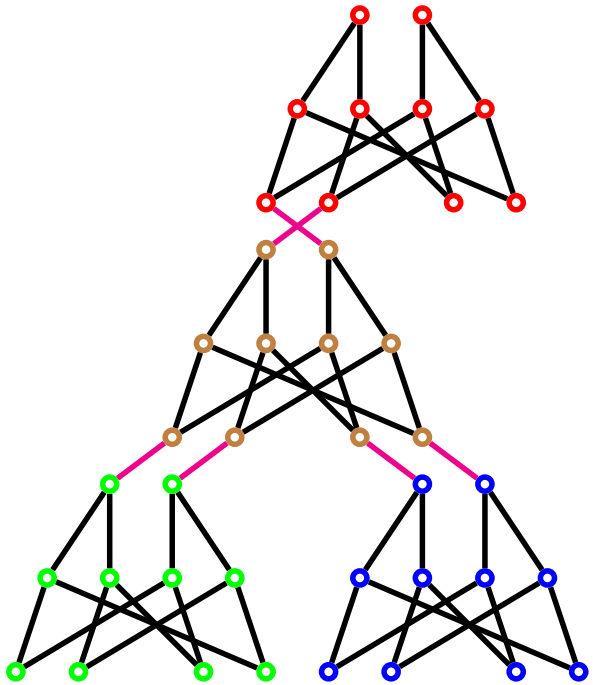

$\tilde{X}(G_n)$

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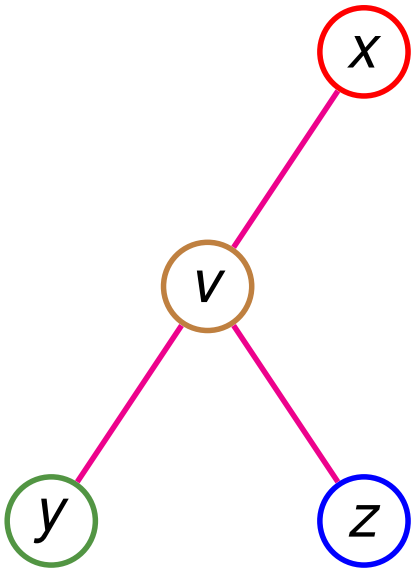

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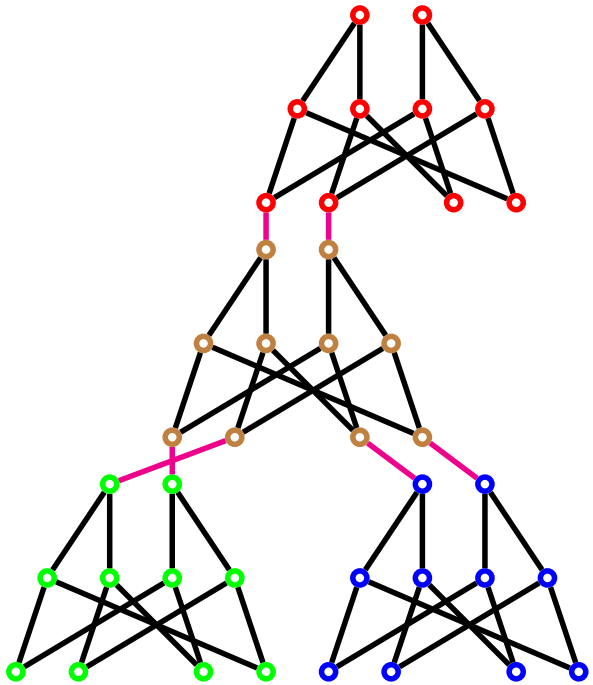

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Prop. Let $X'(G_n)$ be $X(G_n)$ with some number, m, of the magenta edges flipped.

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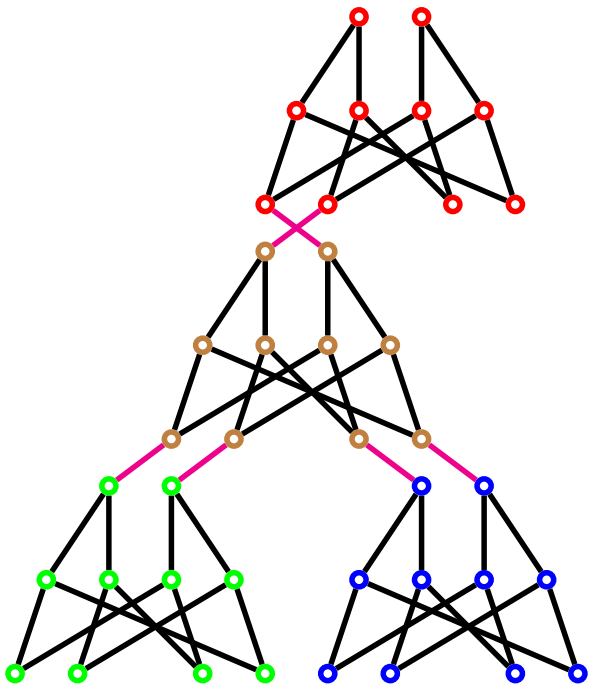

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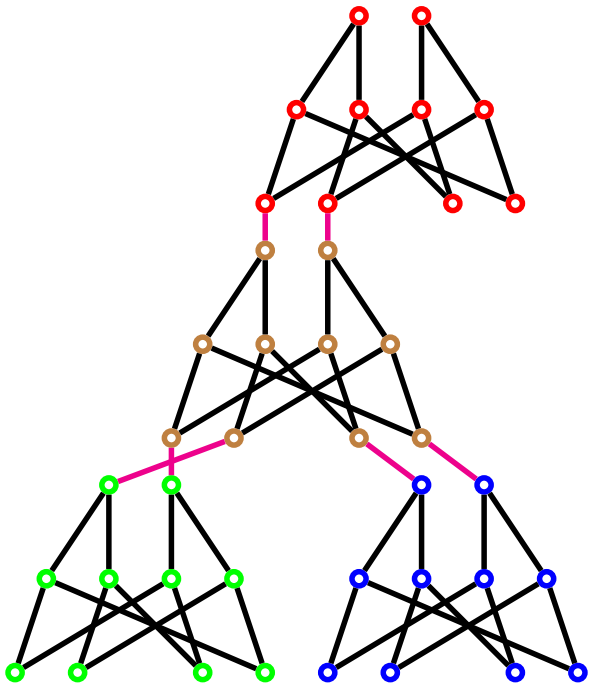

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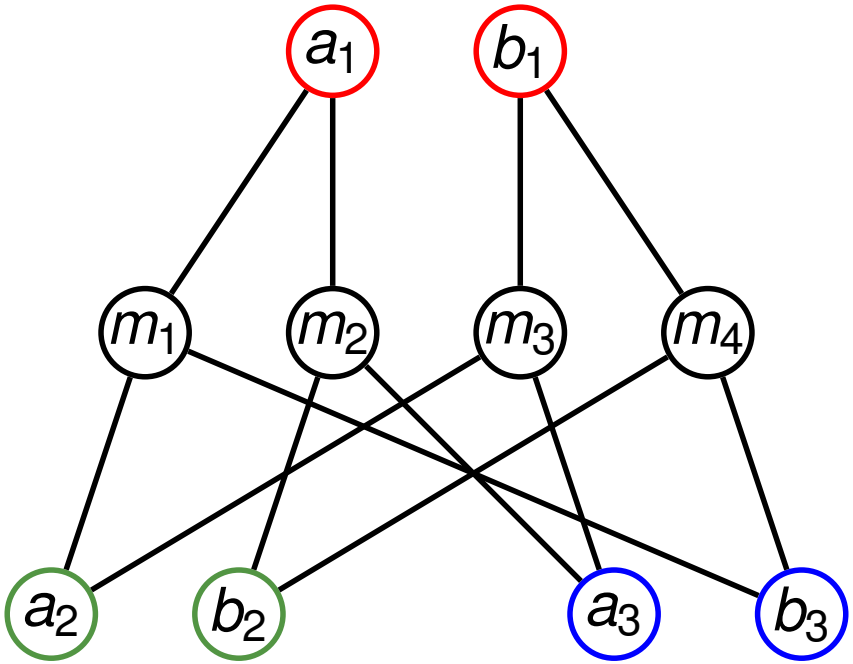

X

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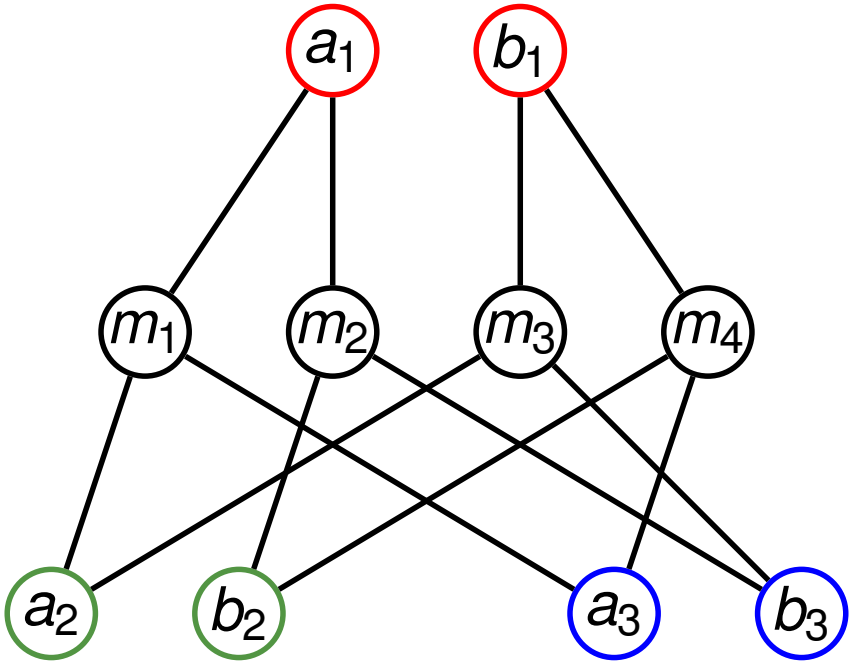

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proof Use the ordering to label boundary pairs a_i, b_i when $a_i \leq b_i$. Then count the number, m, of flips of vertices and edges mod 2. $X'(G) \in \mathrm{CFI}$ iff m is even. □

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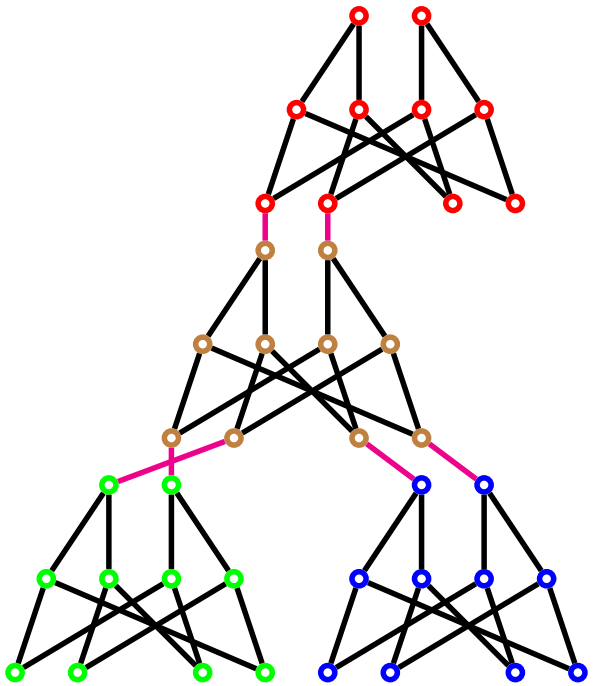

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Thus **Delilah** never loses. □

Recap

We have shown that the linear-time CFI problem is in $\mathrm{OIP} - \mathrm{FPC}$.

Cor. $\Omega(n)$ variables are needed to characterize graphs.

Recent Developments: FPC is Surprisingly Powerful

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Thm. [Anderson, Dawar and Holm] Linear Programming is in FPC.

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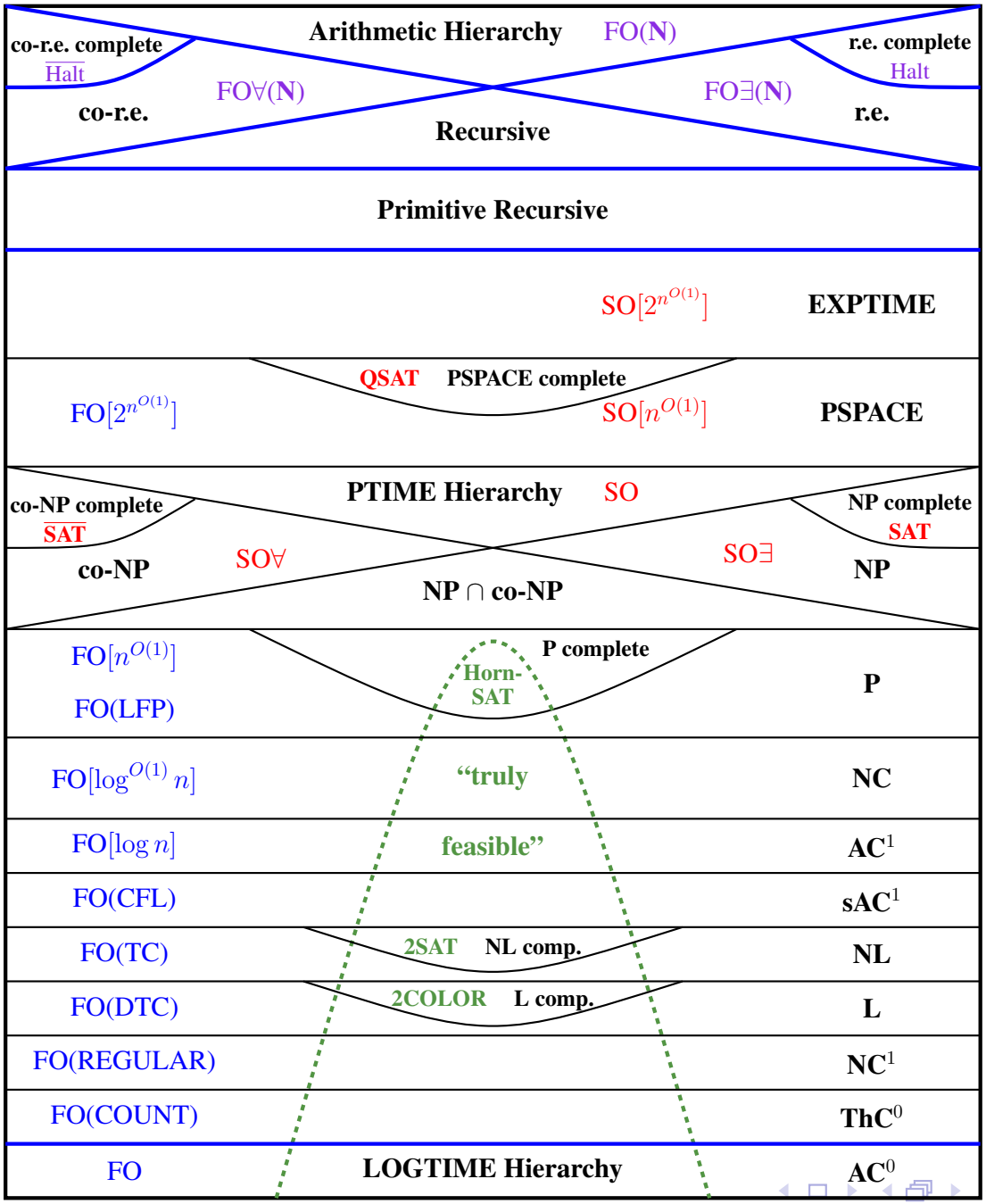
What I want: more natural extension to FPC that adds group theory and characterizes graphs using $O(\log n)$ variables.


Arithmetic Hierarchy
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Recursive
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EXPTIME
QSAT
PSPACE complete
FO[2^{n^{O(1)}}]
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NP
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Horn-SAT
P
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sAC^1
FO(TC)
2SAT
NL comp.
NL
FO(DTC)
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L comp.
L
FO(REGULAR)
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FO
LOGTIME Hierarchy
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