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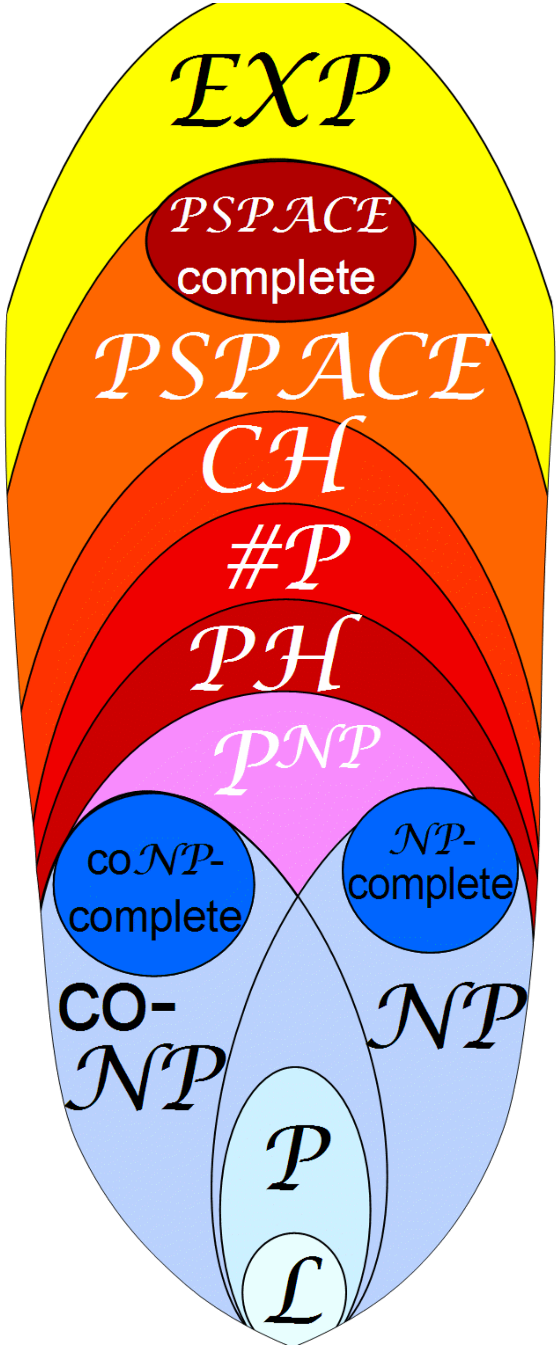

Computability and Complexity

(Decision) Problems

Decidable

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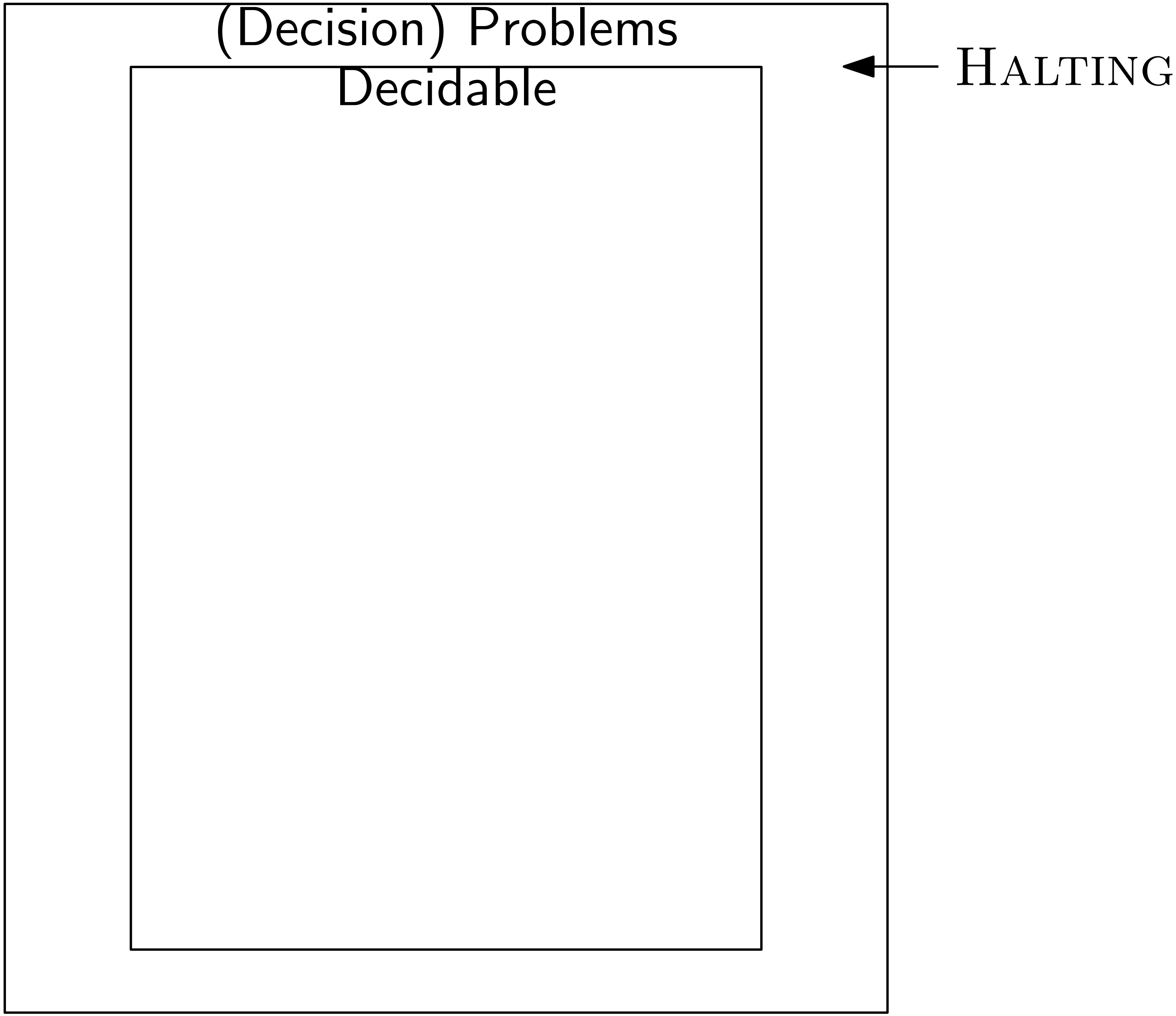

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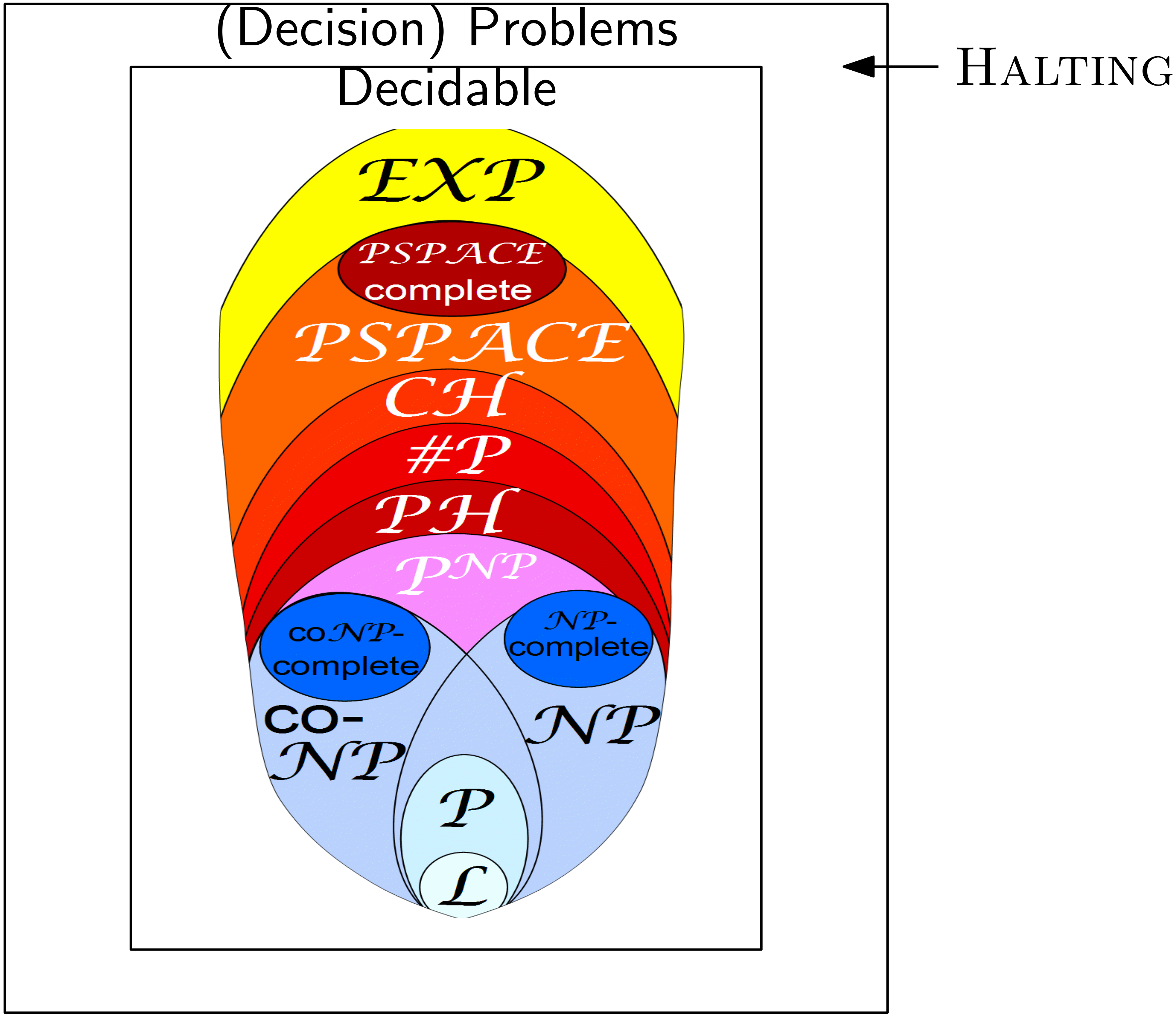

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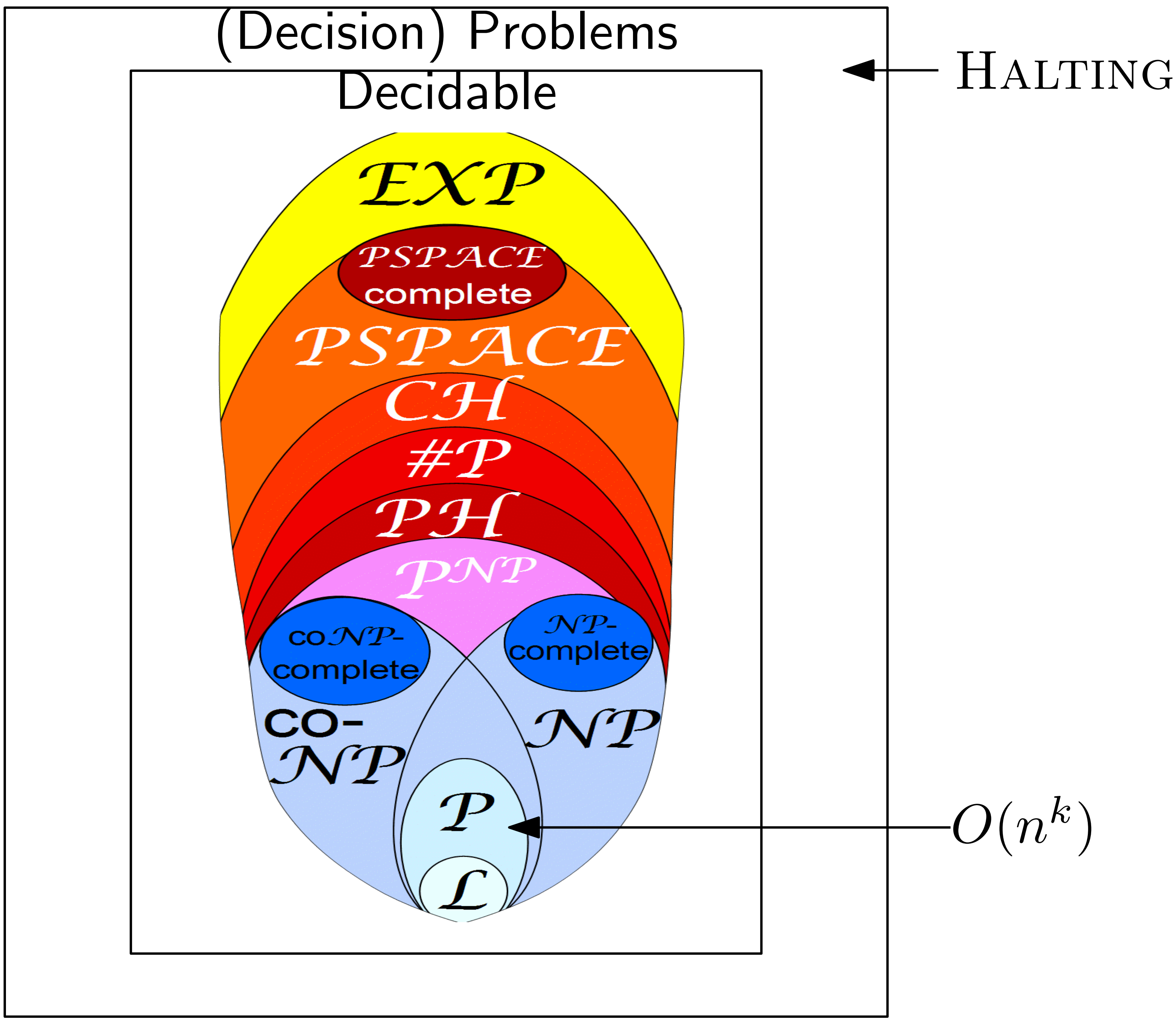

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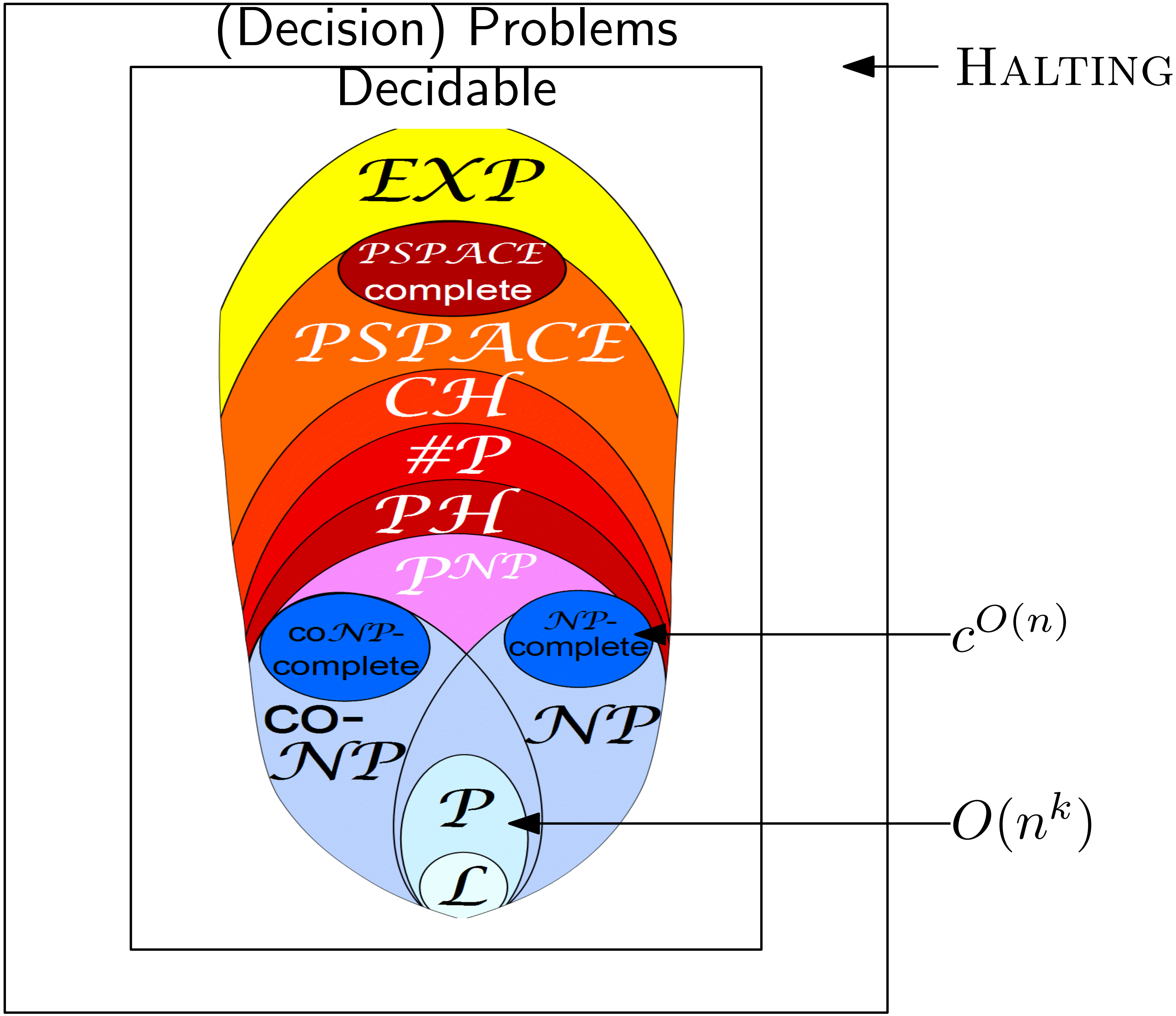

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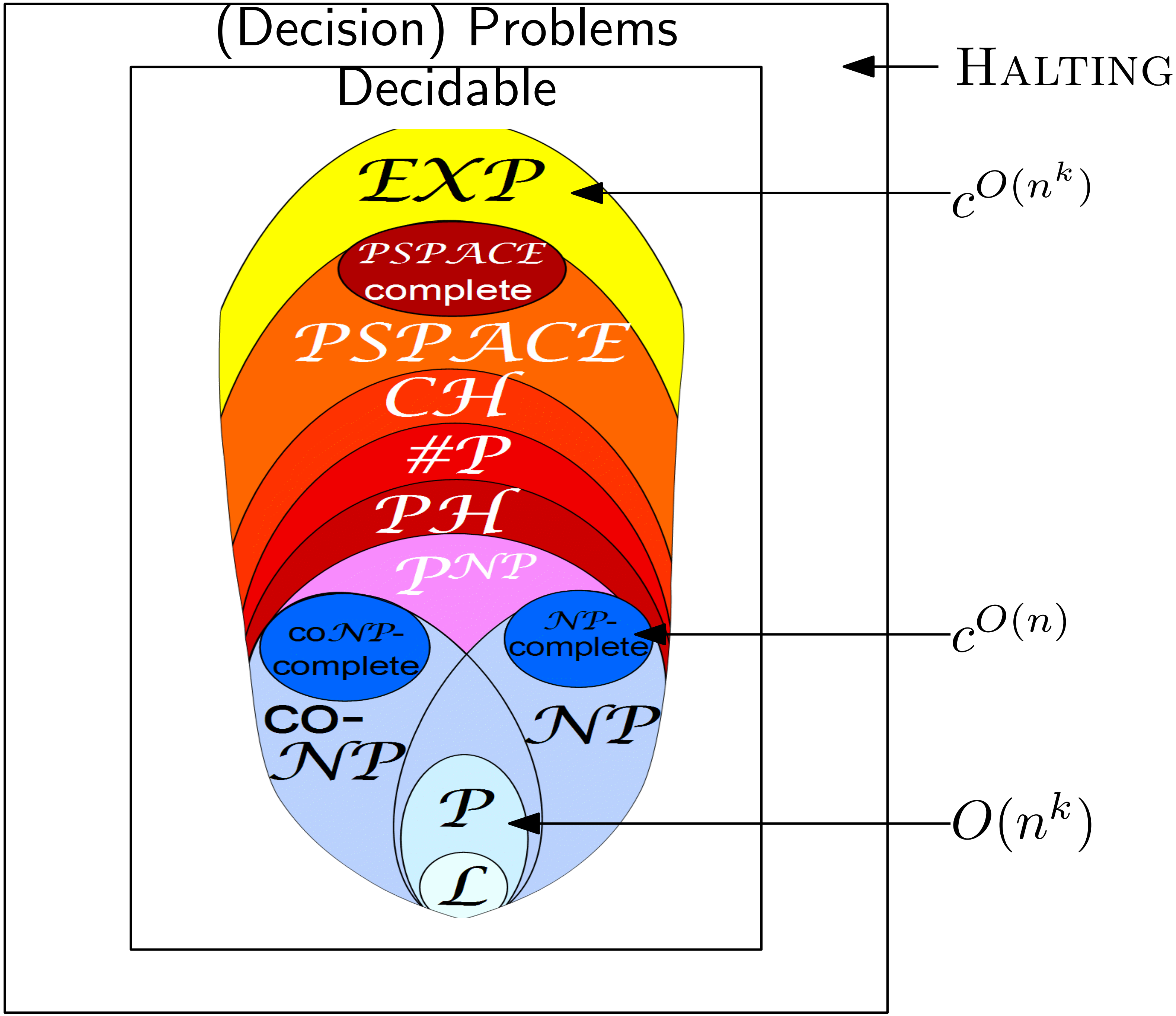

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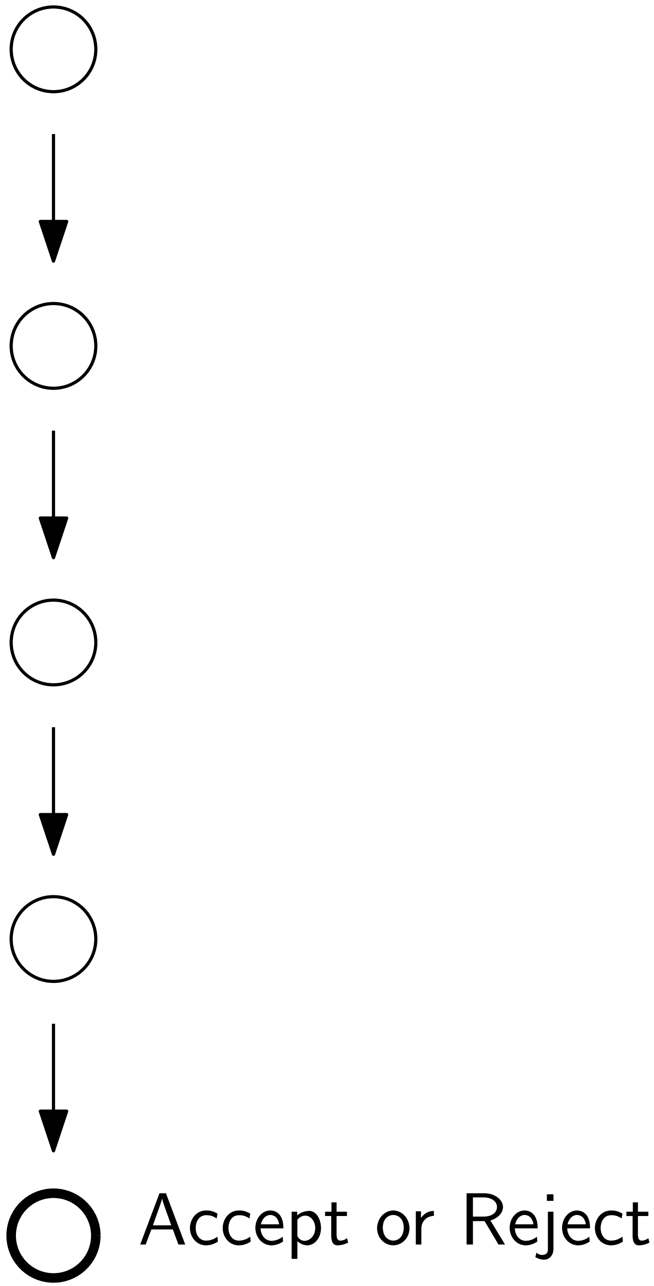

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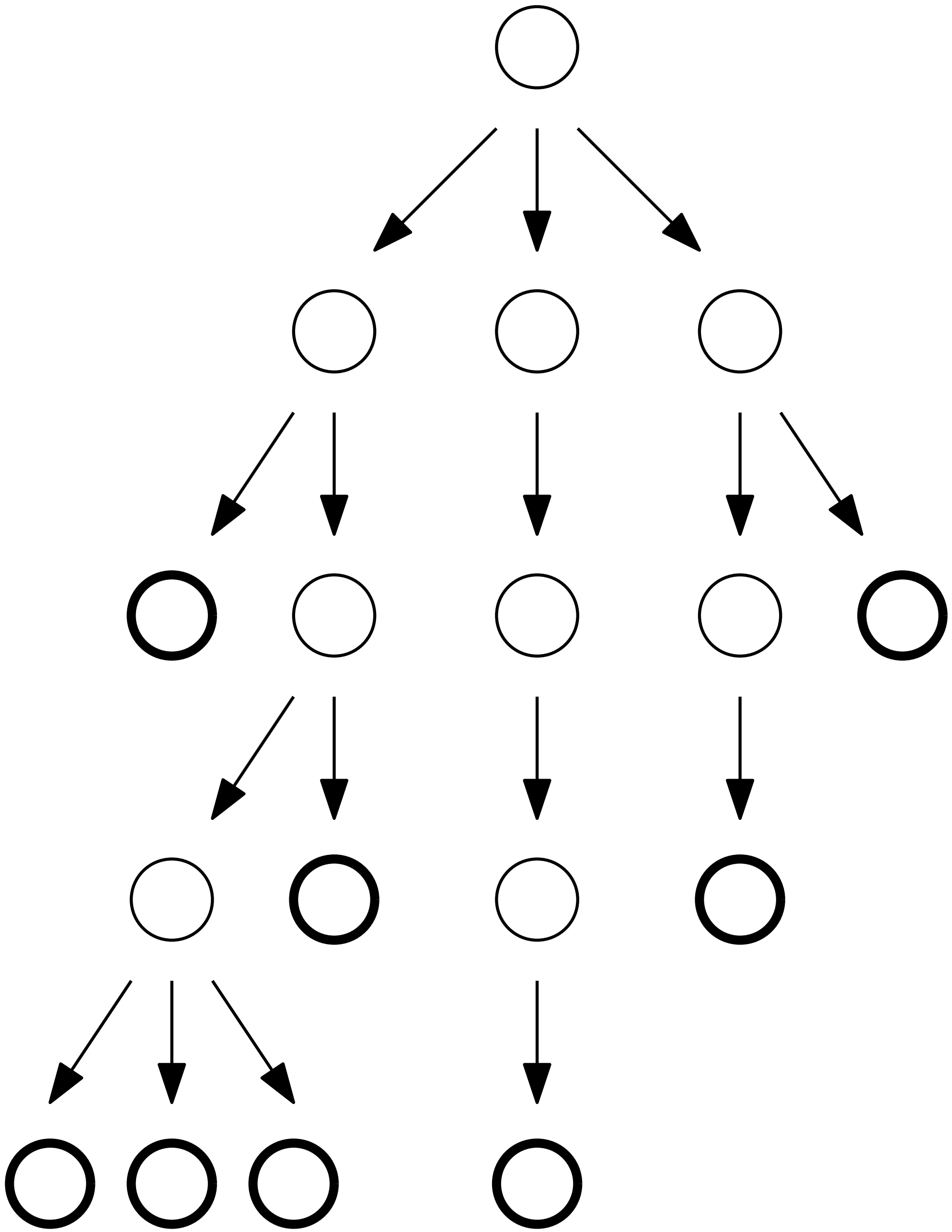

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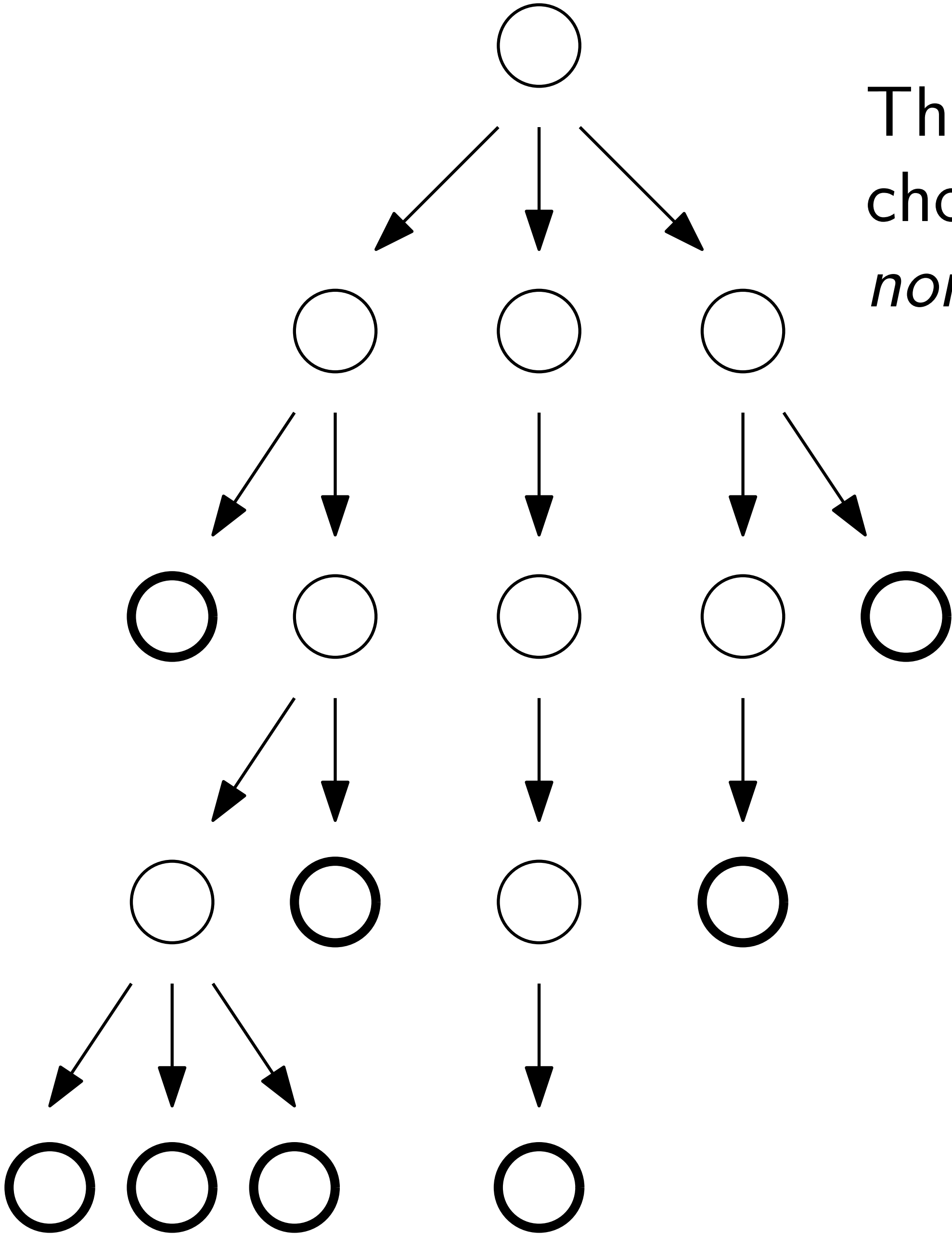

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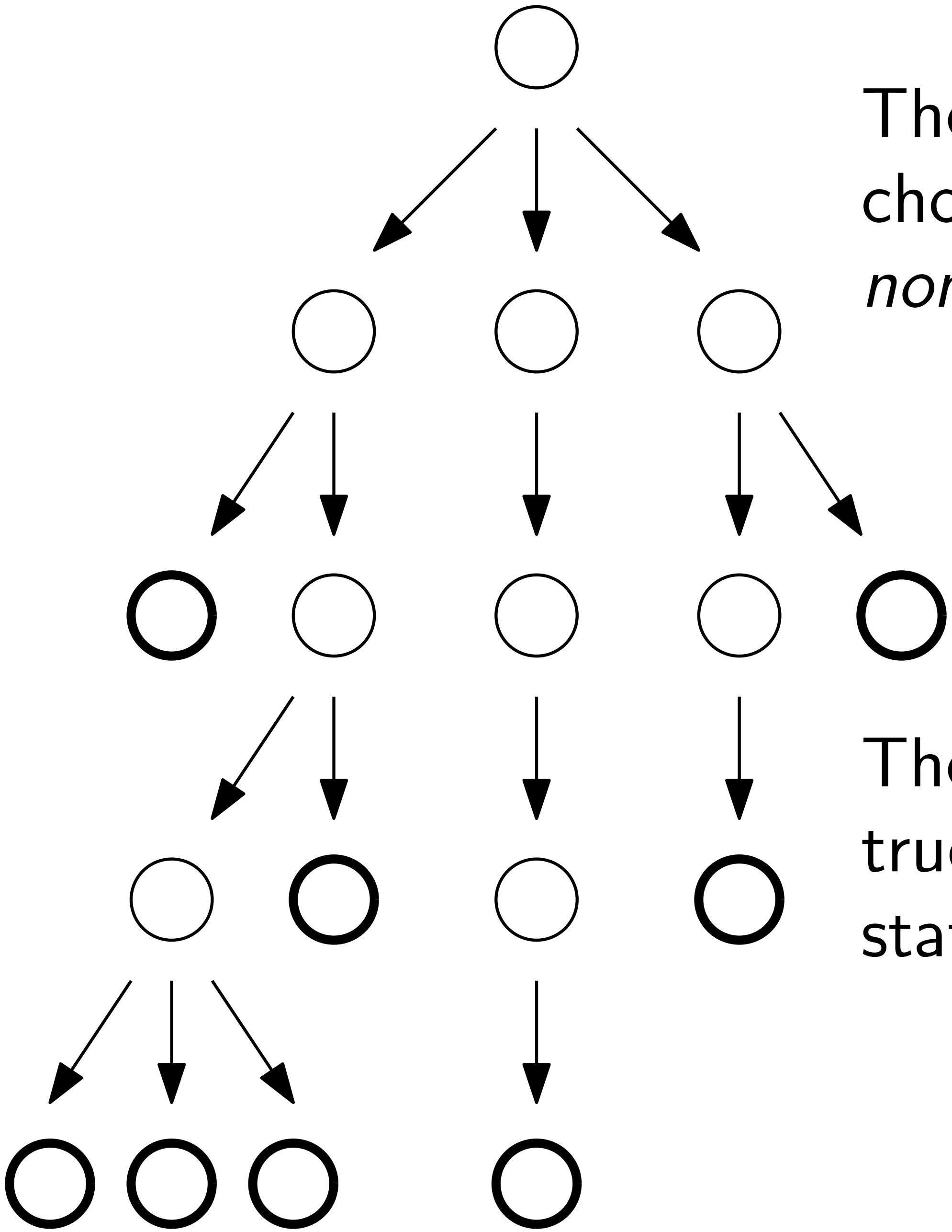

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The program returns true iff one of final states accepts!

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The certificate is a subset of $V(G)$ of size k;
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An instance (G, k) of VertexCover can be converted into an instance $(G, |V(G)| - k)$ of IndependentSet.

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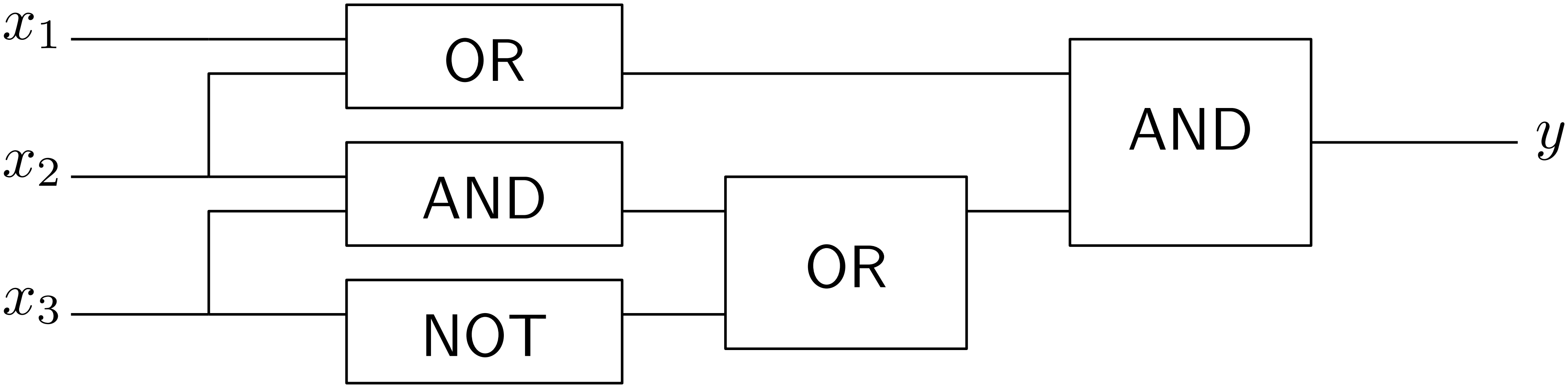

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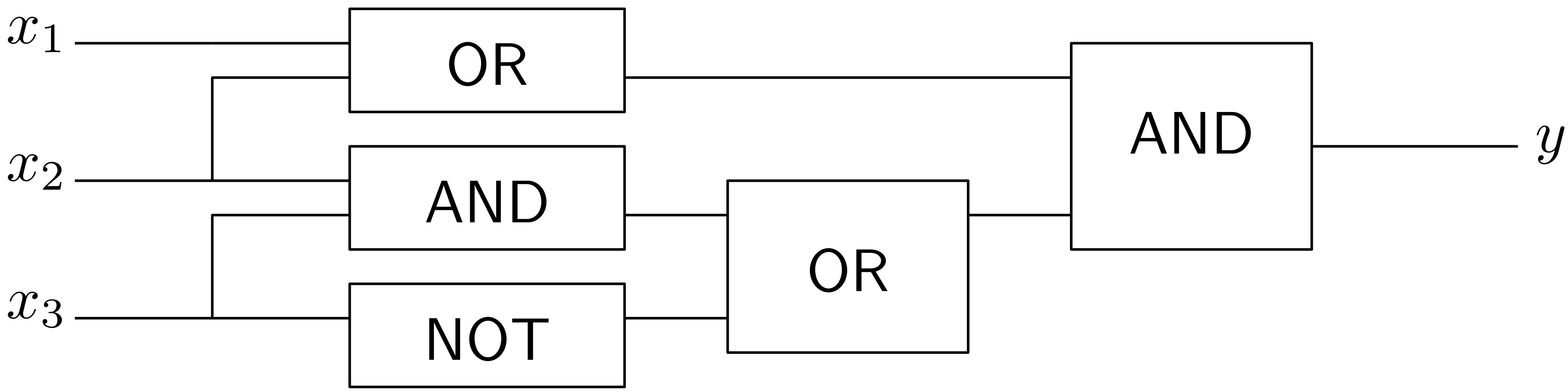

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   Idea: Any turing machine can be imitated by a circuit!

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$\mathrm{3SAT}$: Is a given 3-CNF formula satisfiable?

e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

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e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

- x is a binary variable.
- A literal l is either x or $\neg x$.
- A CNF formula $\phi = C_1 \wedge C_2 \wedge \cdots C_k$ where $C_i = l_{i1} \vee l_{i2} \vee l_{i3}$ for literals l_{ij}.

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Theorem. $\mathrm{3SAT}$ is NP-complete.

1. $\mathrm{3SAT} \in$ NP.
2. $X \leq \mathrm{3SAT}$ for every NP problem X.

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1. $3\mathrm{SAT} \in$ NP.
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$3\mathrm{SAT} \leq \mathrm{CSAT}$ (special case)

$\mathrm{3SAT}$ is NP-complete

$\mathrm{3SAT}$: Is a given 3-CNF formula satisfiable?

e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

Theorem. $\mathrm{3SAT}$ is NP-complete.

1. $\mathrm{3SAT} \in$ NP.
2. $X \leq \mathrm{3SAT}$ for every NP problem X.

We know that CSAT is NP-hard.

$\Rightarrow \mathrm{CSAT} \leq \mathrm{3SAT}$ is sufficient.

$\mathrm{3SAT}$ is NP-complete

$\mathrm{3SAT}$: Is a given 3-CNF formula satisfiable?

e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

Theorem. $\mathrm{3SAT}$ is NP-complete.

1. $\mathrm{3SAT} \in$ NP.

2. $X \leq \mathrm{3SAT}$ for every NP problem X.

Let C be an instance of CSAT.

For each gate, make a variable for its output and simulate the gate.

$3\mathrm{SAT}$ is NP-complete

$3\mathrm{SAT}$: Is a given 3-CNF formula satisfiable?

e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

Theorem. $3\mathrm{SAT}$ is NP-complete.

1. $3\mathrm{SAT} \in$ NP.

2. $X \leq 3\mathrm{SAT}$ for every NP problem X.

Let C be an instance of CSAT.

For each gate, make a variable for its output and simulate the gate.

1) NOT gate

Input value: x_i / Output value: $x_j = \neg x_i$

$\Leftrightarrow (x_i \vee x_j) \wedge (\neg x_i \vee \neg x_j)$

3SAT is NP-complete

3SAT: Is a given 3-CNF formula satisfiable?

e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

Theorem. 3SAT is NP-complete.

1. 3SAT $\in$ NP.

2. $X \leq$ 3SAT for every NP problem X.

Let C be an instance of CSAT.

For each gate, make a variable for its output and simulate the gate.

2) AND gate

Input value: x_i, x_j / Output value: $x_k = x_i \wedge x_j$

$\Leftrightarrow (\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \wedge (x_k \vee \neg x_i \vee \neg x_j)$

3SAT is NP-complete

3SAT: Is a given 3-CNF formula satisfiable?

e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

Theorem. 3SAT is NP-complete.

1. 3SAT $\in$ NP.

2. $X \leq$ 3SAT for every NP problem X.

Let C be an instance of CSAT.

For each gate, make a variable for its output and simulate the gate.

3) OR gate

Input value: x_i, x_j / Output value: $x_k = x_i \vee x_j$

$\Leftrightarrow (x_k \vee \neg x_i) \wedge (x_k \vee \neg x_j) \wedge (\neg x_k \vee x_i \vee x_j)$

3SAT is NP-complete

3SAT: Is a given 3-CNF formula satisfiable?

e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

Theorem. 3SAT is NP-complete.

1. 3SAT $\in$ NP.

2. $X \leq$ 3SAT for every NP problem X.

Let C be an instance of CSAT.

For each gate, make a variable for its output and simulate the gate.

4) To make some input variable x_i true/false, add x_i / $\neg x_i$

3SAT is NP-complete

$\mathrm{3SAT}$: Is a given 3-CNF formula satisfiable?

e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

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Let C be an instance of CSAT.

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5) Replace clauses with 1 or 2 varibles

$x_i \vee x_j \Leftrightarrow (x_i \vee x_j \vee z) \wedge (x_i \vee x_j \vee \neg z)$

$x_i \Leftrightarrow (x_i \vee z \vee w) \wedge (x_i \vee \neg z \vee w) \wedge (x_i \vee z \vee \neg w) \wedge (x_i \vee \neg z \vee \neg w)$

$\mathrm{3SAT}$ is NP-complete

$\mathrm{3SAT}$: Is a given 3-CNF formula satisfiable?

e.g. $\phi = (x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$

Theorem. $\mathrm{3SAT}$ is NP-complete.

1. $\mathrm{3SAT} \in$ NP.

2. $X \leq \mathrm{3SAT}$ for every NP problem X.

   Let C be an instance of CSAT.

   For each gate, make a variable for its output and simulate the gate.

   Claim: C is satisfiable iff ϕ_C is satisfiable.

$\textsc{VertexCover}$ is NP-complete

$S \subseteq V(G)$ is a *vertex cover* for a graph G if every edge of G is incident to at least one vertex of S.

$\textsc{VertexCover}$: given a graph G and integer k, decide if G has a vertex cover of size k.

Theorem. $\textsc{VertexCover}$ is NP-complete.

1. $\textsc{VertexCover} \in$ NP.
2. $3\mathrm{SAT} \leq \textsc{VertexCover}$

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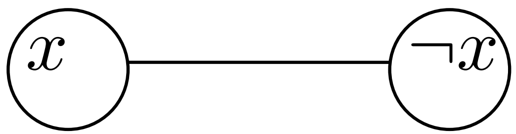
Theorem. $\textsc{VertexCover}$ is NP-complete.

1. $\textsc{VertexCover} \in$ NP.
2. $3\mathrm{SAT} \le \textsc{VertexCover}$

Let ϕ be a 3-CNF with m variables and k clauses.

1) Variable gadget

For each variable x,



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VertexCover: given a graph G and integer k, decide if G has a vertex cover of size k.

Theorem. VertexCover is NP-complete.

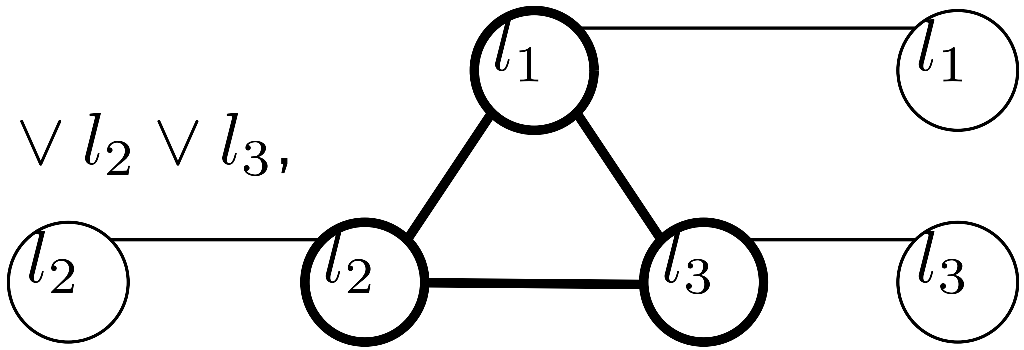
1. VertexCover $\in$ NP.

2. 3SAT $\leq$ VertexCover

Let ϕ be a 3-CNF with m variables and k clauses.

2) Clause gadget

For each clause $C = l_1 \vee l_2 \vee l_3$,



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Theorem. VertexCover is NP-complete.

1. VertexCover $\in$ NP.
2. 3SAT $\leq$ VertexCover

Let ϕ be a 3-CNF with m variables and k clauses.

Claim: ϕ is satisfiable iff G_ϕ has a vertex cover of size $m + 2k$.

SubsetSum is NP-complete

SubsetSum: given a (multi-)set X of integers and an integer s, is there a subset of X whose sum equals to s?

$\textsc{SubsetSum}$ is NP-complete

$\textsc{SubsetSum}$: given a (multi-)set X of integers and an integer s, is there a subset of X whose sum equals to s?

e.g. $X = \{1, 1, 5, 10, 23, 30\}, s = 39$

SubsetSum is NP-complete

SubsetSum: given a (multi-)set X of integers and an integer s, is there a subset of X whose sum equals to s?

e.g. $X = \{1, 1, 5, 10, 23, 30\}, s = 39$ YES! $\{1, 5, 10, 23\}$

$\textsc{SubsetSum}$ is NP-complete

$\textsc{SubsetSum}$: given a (multi-)set X of integers and an integer s, is there a subset of X whose sum equals to s?

Theorem. $\textsc{SubsetSum}$ is NP-complete.

1. $\textsc{SubsetSum} \in \mathsf{NP}$
2. $3\text{SAT} \leq \textsc{SubsetSum}$

SubsetSum is NP-complete

SubsetSum: given a (multi-)set X of integers and an integer s, is there a subset of X whose sum equals to s?

Theorem. SubsetSum is NP-complete.

1. SubsetSum $\in$ NP
2. 3SAT $\leq$ SubsetSum

Certificate: a set of integers Y

Certifer: 1) Y is a subset of X and 2) sum of Y equals to s

SubsetSum is NP-complete

SubsetSum: given a (multi-)set X of integers and an integer s, is there a subset of X whose sum equals to s?

Theorem. SubsetSum is NP-complete.

1. SubsetSum $\in$ NP.
2. $3\mathrm{SAT} \leq$ SubsetSum

Let ϕ be a 3-CNF with m variables and k clauses.

Construct integers t_i, f_i of $m+k$ digits for each variable x_i

Construct integers a_j, b_j of $m+k$ digits for each clause C_j

m digits correspond to T/F assignment for each variable.

t_i, f_i have 1 for i-th digit.

SubsetSum is NP-complete

SubsetSum: given a (multi-)set X of integers and an integer s, is there a subset of X whose sum equals to s?

Theorem. SubsetSum is NP-complete.

1. SubsetSum $\in$ NP.

2. 3SAT $\leq$ SubsetSum

   Let ϕ be a 3-CNF with m variables and k clauses.

   Construct integers t_i, f_i of $m+k$ digits for each variable x_i

   Construct integers a_j, b_j of $m+k$ digits for each clause C_j

   k digits correspond to satisfiability for each clause.

   t_i/f_i has 1 for $(m+j)$-th digit if $x_i/\neg x_i$ appear in C_j.

   $a_j = b_j$ have 1 for $(m+j)$-th digit.

$\textsc{SubsetSum}$ is NP-complete

$\textsc{SubsetSum}$: given a (multi-)set X of integers and an integer s, is there a subset of X whose sum equals to s?

Theorem. $\textsc{SubsetSum}$ is NP-complete.

1. $\textsc{SubsetSum} \in$ NP.
2. $3\mathrm{SAT} \leq \textsc{SubsetSum}$

Let ϕ be a 3-CNF with m variables and k clauses.

Claim: ϕ is satisfiable iff there is a subset of $\{t_1, f_1, \cdots, t_m, f_m, a_1, b_1, \cdots, a_k, b_k\}$ of sum:

$$\underbrace{11\cdots1}_{m}\underbrace{33\cdots3}_{k}$$

DNF-SAT

DNF-SAT: Is a given DNF formula satisfiable?

e.g. $\phi = (x_1 \wedge x_2 \wedge x_3) \vee (\neg x_1 \wedge \neg x_2) \vee \neg x_3$

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- x is a binary variable.
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- DNF-SAT $\in$ P.

DNF-SAT

DNF-SAT: Is a given DNF formula satisfiable?

e.g. $\phi = (x_1 \wedge x_2 \wedge x_3) \vee (\neg x_1 \wedge \neg x_2) \vee \neg x_3$

- DNF-SAT $\in$ P.
- 3SAT $\leq$ DNF-SAT

DNF-SAT

DNF-SAT: Is a given DNF formula satisfiable?

e.g. $\phi = (x_1 \land x_2 \land x_3) \lor (\neg x_1 \land \neg x_2) \lor \neg x_3$

- DNF-SAT $\in$ P.
- 3SAT $\leq$ DNF-SAT

  Let ϕ be a 3-CNF with m variables and k clauses.

  We can convert CNF into DNF as follows:

  $(a \lor b \lor c) \land (d \lor e \lor f)$
  $= (a \land d) \lor (a \land e) \lor (a \land f) \lor (b \land d) \lor (b \land e) \lor (b \land f) \lor$
  $(c \land d) \lor (c \land e) \lor (c \land f)$

DNF-SAT

DNF-SAT: Is a given DNF formula satisfiable?

e.g. $\phi = (x_1 \wedge x_2 \wedge x_3) \vee (\neg x_1 \wedge \neg x_2) \vee \neg x_3$

- DNF-SAT $\in$ P.
- 3SAT $\leq$ DNF-SAT

  Let ϕ be a 3-CNF with m variables and k clauses.

  We can convert CNF into DNF as follows:

  $$(a \vee b \vee c) \wedge (d \vee e \vee f)$$
  $$= (a \wedge d) \vee (a \wedge e) \vee (a \wedge f) \vee (b \wedge d) \vee (b \wedge e) \vee (b \wedge f) \vee$$
  $$(c \wedge d) \vee (c \wedge e) \vee (c \wedge f)$$

NOT a polynomial-time reduction!