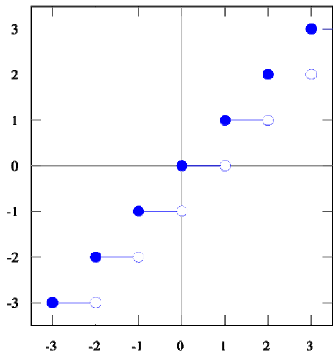
Two Effects in Real Computation

a) Multivalued 'functions'



Example (Archimedian Reals):
Given $x \in \mathbb{R}$,
return <u>some</u> integer upper bound.

Example (Fundamental Theorem of Algebra):
Given $a_0,\ldots a_{d-1} \in \mathbb{C}$, return roots $x_1,\ldots x_d \in \mathbb{C}$ of
$a_0 + a_1 \cdot X + \ldots + a_{d-1} \cdot X^{d-1} + X^d \in \mathbb{C}[X]$ incl. multiplicities

up to permutation [Specker'67]

b) Discrete 'advice'

Example matrix diagonalization: given $A \in \mathbb{R}^{d \cdot (d-1)/2}$,
return a basis of eigenvectors — discontinuous:

Thm: Computable knowing $|\sigma(A)|$. $\varepsilon \cdot \begin{pmatrix} \cos(2/\varepsilon) & \sin(2/\varepsilon) \\ \sin(2/\varepsilon) & -\cos(2/\varepsilon) \end{pmatrix}$

Overview

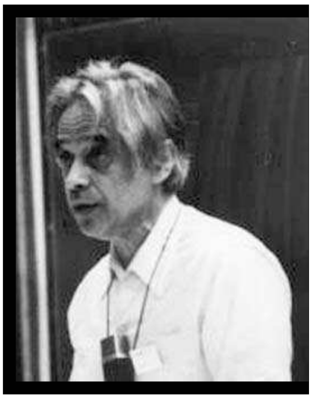
<u>2. Real Complexity Theory:</u>

- numbers, sequences, limits
- real functions and continuity
- maximizing polytime functions
- integration and solving ODEs
- solving Poisson's PDE
- analytic functions, enrichment
- practical implementation/iRRAM
- parameterized Gevrey's Hierarchy

Computable Real Numbers

Theorem: For $r \in \mathbb{R}$, the following are equivalent:

Call $r \in \mathbb{R}$ computable if



a) r has a computable binary expansion

b) There is an algorithm printing, on input $n \in \mathbb{N}$, some $a \in \mathbb{Z}$ with $|r - a/2^n| \leq 2^{-n}$.

c) There is an algorithm producing three sequences $(a_n),(b_n),(c_n) \subseteq \mathbb{Z}$ with $|r - a_n/b_n| \leq 1/c_n \to 0$

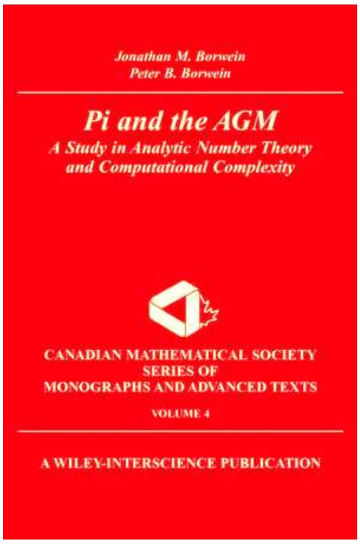
Ernst Specker (1949): (c) $\Leftrightarrow$ *Halting problem* plus (d)

d) There is an algorithm printing $(q_n) \subseteq \mathbb{Q}$ with $q_n \to r$.

Polytime-Computable Reals

Example: The following are polytime computable

- sum, product, and reciproce of polytime-computable reals
- every algebraic real
- every real root of a polynomial polytime-computable coefficients
- some transcendental reals such as $e = 2.718..$ or π.



Definition: $r \in \mathbb{R}$ is polytime computable iff some Turing machine can print within time polynomial in n a sequence $(a_n) \subseteq \mathbb{Z}$ in binary s.t. $|r - a_n/2^n| \leq 2^{-n}$.

Polytime-Computable Functions

Definition: A real sequence $(r_j)\subseteq\mathbb{R}$ is polytime computable iff a Turing machine can print in time $\leq\text{poly}(n)$ a sequence $(a_n)\subseteq\mathbb{Z}$ with $|r_j - a_{\langle j,m\rangle}/2^m|\leq 2^{-m}$;

equivalently: given j and m print, within $\leq\text{poly}(j+m)$ steps, some $a\in\mathbb{Z}$ such that $|r_j - a/2^m|\leq 2^{-m}$

sequence index enters in unary

Definition: Computing function $f:[0;1]\to\mathbb{R}$ in time $t(n)$ means to print, given $\underline{a}\subseteq\mathbb{Z}$ with $|x - a_m/2^m|\leq 2^{-m}$, $(b_n)\subseteq\mathbb{Z}$ with $|f(x) - b_n/2^n|\leq 2^{-n}$ after at most $t(n)$ steps.

Definition: $r\in\mathbb{R}$ is polytime computable iff some Turing machine can print within time polynomial in n a sequence $(a_n)\subseteq\mathbb{Z}$ in binary s.t. $|r - a_n/2^n|\leq 2^{-n}$.

Examples of Polytime Functions

- Behaviour is undefined for $\underline{x}\notin\text{dom}(f)$ or other $\underline{a}$.
- Runtime may depend only on output precision n.

Exercise: $\exp:\mathbb{R}\to\mathbb{R}$ is not computable in bounded time, but on $[-2^k;k]$ computable in time $\text{poly}(n+k)$.

Theorem: Let $(c_j)\subseteq\mathbb{C}\approx\mathbb{R}^2$ be polytime computable, $R:=1/\limsup_j |c_j|^{1/j}>0$, and $0<r<R$. Then $\{z\in\mathbb{C} : |z|\leq r\} \ni z \to \sum_j c_j\cdot z^j$ is polytime computable.

Def: Computing $f:\subseteq\mathbb{R}^d\to\mathbb{R}^e$ in time $t(n)$ means to print, given $(\underline{a})_m \subseteq\mathbb{Z}^d$ s.t. $|\underline{x} - \underline{a}_m/2^m|\leq 2^{-m}$ for $\underline{x}\in\text{dom}(f)$, $(\underline{b})_n \subseteq\mathbb{Z}^e$ such that $|f(\underline{x}) - \underline{b}_n/2^n|\leq 2^{-n}$ within $t(n)$ steps.

Computing and Continuity

- Behaviour is undefined for $\underline{x} \notin \mathrm{dom}(f)$ or other $\underline{a}$.
- Runtime may depend only on output precision n.

Theorem: If $f:\subseteq\mathbb{R}\to\mathbb{R}$ is computable in time $t(n)$, then $\mu(n):=t(n+1)+1$ is a <u>modulus of continuity</u> of f.

$$x,x'\in\mathrm{dom}(f),\ |x-x'|\le 2^{-\mu(n)}\ \Rightarrow\ |f(x)-f(x')|\le 2^{-n}.$$

Theorem: Suppose differentiable $f:[0;1]\to\mathbb{C}$ is polytime and f' has a polynomial modulus of continuity. Then f' is again polytime computable.

Def: Computing $f:\subseteq\mathbb{R}^d\to\mathbb{R}^e$ in time $t(n)$ means to print, given $(\underline{a})_m\subseteq\mathbb{Z}^d$ s.t. $|\underline{x}-\underline{a}_m/2^m|\le 2^{-m}$ for $\underline{x}\in\mathrm{dom}(f)$, $(\underline{b})_n\subseteq\mathbb{Z}^e$ such that $|f(\underline{x})-\underline{b}_n/2^n|\le 2^{-n}$ within $t(n)$ steps.

More Polytime Functions

- Behaviour is undefined for $\underline{x} \notin \mathrm{dom}(f)$ or other $\underline{a}$.
- Runtime may depend only on output precision n.

Theorem: Let $f:[-1;0]\to\mathbb{C}$, $g:[0;1]\to\mathbb{C}$ be polytime with $f(0)=g(0)$. Then $f\cup g:[-1;1]\to\mathbb{C}$ is polytime, too

Theorem: If $f:[0;1]\to\mathbb{R}$ is computable, then so within bounded time $t(n)$ for some $t:\mathbb{N}\to\mathbb{N}$

Theorem: Let $f:[0;1]\to[0;1]$ be polytime and bijective s.t. f^{-1} has a polynomial modulus of continuity. Then $f^{-1}:[0;1]\to[0;1]$ is polytime, too.

$(\underline{b})_n\subseteq\mathbb{Z}^e$ such that $|f(\underline{x})-\underline{b}_n/2^n|\le 2^{-n}$ within $t(n)$ steps.

$\mathcal{P}$ vs $\mathcal{NP}$ Millennium Problem

Def: Turing Machine $\mathcal{M}$ decides set $L\subseteq\{0,1\}^*$ if
- on inputs $\underline{x}\in L$ prints 1 and terminates,
- on inputs $\underline{x}\notin L$ prints 0 and terminates.

Def: $\mathcal{M}$ runs in polynomial time if $\exists p\in\mathbb{N}[N]$:
$\mathcal{M}$ on input $\underline{x}\in\{0,1\}^n$ makes at most $p(n)$ steps

Def: $L\subseteq\{0,1\}^*$ is verifiable in polynomial time if
$L = \{\ \underline{x}\in\{0,1\}^n \mid n\in\mathbb{N},\ \exists\underline{y}\in\{0,1\}^{q(n)} :\ \langle\underline{x},\underline{y}\rangle\in V\ \},\ \ V\in\mathcal{P}$

$\mathcal{P}$:= { $L\subseteq\{0,1\}^*$ decidable in polynomial time }
$\subseteq$ $\mathcal{NP}$ **:= { L verifiable in polynomial time }**
$\subseteq$ $\mathcal{PSPACE}$:= { L decidable in polyn. space }
$\subseteq$ $\mathcal{EXP}$:= { L decidable in exponential time }

Complexity of 1D Maximization

Fix polytime $f:[0;1]\to[0;1]$ ($\Rightarrow$ continuous)

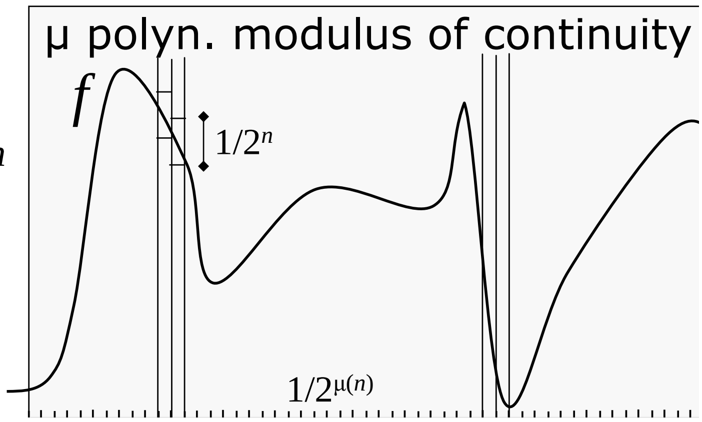
$\mathrm{Max}(f): [0;1]\ni x\to\max\{\ f(t): t\leq x\ \}$
is computable in exponential time / polyn.space
in polynom. time, provided that $\mathcal{P}=\mathcal{NP}$ holds:

Let $\Phi_n:\subseteq\mathbb{N}\to\mathbb{N}$ be computable in time polyn. in n
s.t. $\ |\,f(a/2^{\mu(n)}) - \Phi_n(a)\,| \leq 2^{-n}$ for all $a\in\{0,\ldots,2^{\mu(n)}\text{-}1\}$

and employ

$\{\ (2^n,b,c)\mid\ b<2^{\mu(n)},\ c<2^n$
$\exists a\leq b: \Phi_n(a)\geq c\ \}\in\mathcal{NP}$

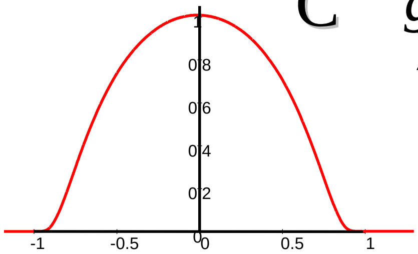
for bisection w.r.t. c.



'Max is $\mathcal{NP}$-hard'

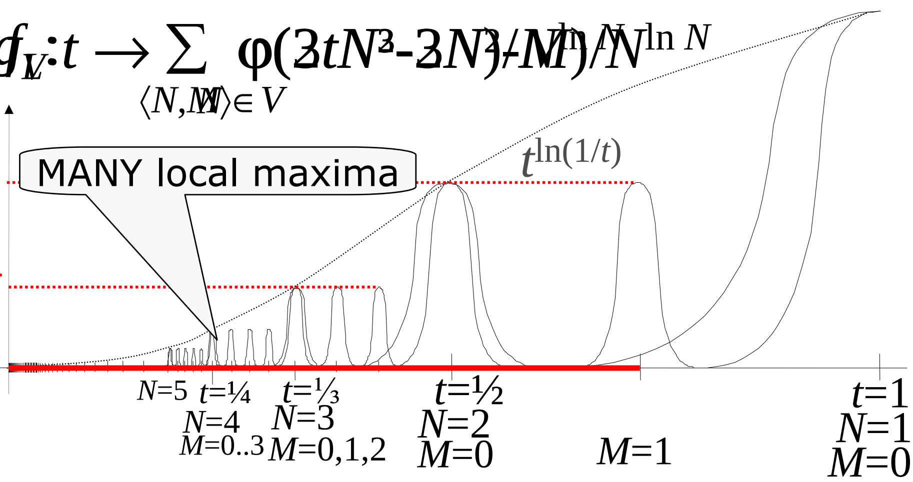
$\mathcal{NP} \ni L = \{ N\in\mathbb{N} \mid \exists M<N: \langle N,M\rangle\in V \},\ V\in\mathcal{P}$ polytime

C∞ $f_V: t \to \sum_{\langle N,M\rangle\in V} \varphi(3tN^3-3N^3-M)/N^{\ln N}$



$\varphi(t) = \exp(-t^2/1-t^2)$

C∞ 'pulse' function
polytime computable



To every $L\in\mathcal{NP}$ there exists a polytime computable C∞ function $g_L:[0;1]\to\mathbb{R}$ s.t.: $[0;1]\ni x\to\max g_L|_{[0;x]}$ again polytime iff $L\in\mathcal{P}$

Computational Complexity

Fix polytime $f:[0;1]\to[0;1]$ ($\Rightarrow$ continuous)

[Friedman&Ko'82ff]

- Max: $f \to \mathrm{Max}(f): x \to \max\{ f(t): t \leq x \}$
  Max(f) computable in exponential time;
  polyn.time-computable iff $\mathcal{P}=\mathcal{NP}$

even when restricting to $f\in C^\infty$

- $\int: f \to \int f: (x \to \int_0^x f(t)\, dt)$
  $\int f$ computable in exponential time;
  polyn.time-computable iff $\mathcal{P}=\#\mathcal{P}$

polytime if f is analytic

- odesolve: $C^1([0;1]\times[-1;1])\ni f\to z$: $\dot{z}(t)=f(t,z)$, $z(0)=0$.
  $\mathcal{PSPACE}$-"complete"

[Kawamura,Ota,Rösnick,Z. *Logical Meth.Comp.Sci.*'14]

- Solution to Poisson Eq. is classical and $\#\mathcal{P}$-"complete"

$\Delta u = f$ on $B_2(\mathbf{0},1)$

[Kawamura, Steinberg, Z.'13; full version MSCS'16]