Martin Ziegler
Jiwon Park

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CS700

Fall 2016, Assignment #3

PROBLEM 3 (1+1+2+1+1+1P)**:**
Abbreviate $|\vec{x}| := n$ for $\vec{x} \in \{0,1\}^n$ and $(x_1,\ldots,x_n)_j := x_j$; also consider integers encoded in binary.

a) Prove that the following problem is in $\mathcal{NP}$ for every $V \in \mathcal{P}$:

$$V' \;:=\; \big\{ x \in \mathbb{N} \;\big|\; \exists y \in \mathbb{N}:\; y \leq x \,\wedge\, (x,y) \in V \big\}$$

b) Prove that there exists some $V \in \mathcal{P}$ such that V' is $\mathcal{NP}$-hard.

c) Prove that a function $f : \{0,1\}^* \to \{0,1\}^*$ is computable in polynomial time iff it holds (i) $|f(\vec{x})| \leq p(|\vec{x}|)$ for some polynomial p and every $\vec{x}$, and (ii) $\{(\vec{x},j) \;\big|\; f(\vec{x})_j = 1\} \in \mathcal{P}$.

d) Suppose $V \in \mathcal{P} = \mathcal{NP}$ Is the following function computable in polynomial time?

$$\mathbb{N} \ni x \mapsto \operatorname{Card}\{y \in \mathbb{N} \;\big|\; y \leq x \,\wedge\, (x,y) \in V\} \in \mathbb{N}$$

e) Prove that every decision problem in $\mathcal{NP}$, as well as the function from (d),
can be solved/computed using a polynomial amount of memory (bits).

f) Prove that an algorithm using at most $s(n) \geq n$ bits of memory on binary inputs of length n before terminating, can make at most $2^{\mathcal{O}(s(n))}$ steps.

Let $f : X \to Y$ be a function between metric spaces (X,d) and (Y,e). Recall that a *modulus of continuity* of f is a mapping $\mu : \mathbb{N} \to \mathbb{N}$ satisfying: $\quad d(x,x') \leq 2^{-\mu(n)} \quad\Rightarrow\quad e\big(f(x),f(x')\big) \leq 2^{-n}$. Also, f is *Hölder-continuous of exponent* $\alpha > 0$ if there exists some L such that $e\big(f(x),f(x')\big) \leq L \cdot d(x,x')^{\alpha}$ for all $x,x' \in \operatorname{dom}(f)$. *Lipschitz-continuous* means Hölder-continuous of exponent 1.

PROBLEM 4 (1+1+1+1+1+1+1+1P)**:**

a) Prove that every $f \in \mathcal{C}^1[0;1]$ (i.e. continuously differentiable $f : [0;1] \to \mathbb{R}$) is Lipschitz-continuous.

b) Prove that every Lipschitz-continuous $f : [0;1] \to \mathbb{R}$ has a modulus of continuity $\mu(m) = m + c$ for some $c \in \mathbb{N}$;

c) and vice versa: every $f : [0;1] \to \mathbb{R}$ with modulus of continuity $\mu(m) = m + c$ for some $c \in \mathbb{N}$ is Lipschitz-continuous.

d) Prove that every Hölder-continuous $f : [0;1] \to \mathbb{R}$ has a modulus of continuity $\mu(m) = a \cdot m + c$ for some $a, c \in \mathbb{N}$;

e) and vice versa.

f) Prove that $f : [0;1] \ni x \mapsto \sqrt{x} \in [0;1]$ is Hölder-continuous but not Lipschitz.

g) Sketch/plot the function $g : [0;1] \ni x \mapsto 1/\ln(e/x) \in [0;1]$.
Prove that it is continuous with an exponential, but no polynomial, modulus of continuity.

h) Prove that $g \circ g$ has no exponential, but a doubly exponential, modulus of continuity.